特異空間の変分法

課題番号 09640086

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平成１２年３月

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（福島大学教育学部）

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研究発表

石井博行　A generalization of eigenvalue problems for system of second order linear differential equations,
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大橋勝弘　On the functional central limit theorem for Walsh series with general gaps,
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論文目次

1. MATSUI, Akinori, Variations of Graphs in Riemannian manifolds, preprint
2. MATSUI, Akinori, Distributions on Graphs and Walks, preprint
3. OHASHI,Katsuhiro, On the functional cental limit theorem for Walsh series with general gaps
4. ISHII,Hiroyuki, A generalization of eigenvalue problems for system of second order linear differential equations

VARIATIONS OF GRAPHS IN RIEMANNIAN MANIFOLDS

AKINORI MATSUI

1 TENSION VECTORS AND VARIATION

In this paper, we study the variation of a graph embedded in a Riemannian manifold. Let each edge of a graph have the property of springs on tension. Suppose that a graph is embedded in a Riemannian manifold such that each edge is geodesical. On this situation, we will introduce the notion of a tension vector at each vertex of a graph and that of a tension Jacobi field on a graph. If an expanded graph moves in the Euclidian space by the influence of tension, then it moves to such direction that the sum of the lengths of its tension vectors. Then we propose the following.
PROPOSAL : *The sum of the lengths of tension vectors decreases if the graph moves along to the tension Jacobi field.*
If the ambient Riemannian manifold has negative curvature, this proposal is true (Corollary 1.3), but this proposal is not always true. In Section 3, we construct examples which are not satisfied with this proposal. In Section 1, we formulate the variation formula of tension vectors. We prove this in Section 2.

Let G be a graph. For each edge α of G and a positive number $\iota(\alpha)$, we call a homeomorphism $\iota_\alpha : \alpha \to [0, \iota(\alpha)]$ the *length function* of α and we call $\iota(\alpha)$ the length of α. If, for each edge α of G, a length functions of α is chosen, we call G a *graph with length.* For each vertex p and each edge α with $p \prec \alpha$, we define $\iota_{p\alpha} : \alpha \to [0, \iota(\alpha)]$ by
$\iota_{p\alpha}(x) = \iota_\alpha(x)$ if $\iota_\alpha(p) = 0$ and $\iota_{p\alpha}(x) = \iota(\alpha) - \iota_\alpha(x)$ if $\iota_\alpha(p) = \iota(\alpha)$.
We call $\iota_{p\alpha}$ the *length function* of α from p. Let G be a graph with length and M a Riemannian manifold. For a mapping $f : G \to M$, we put $f_\alpha = f \circ \iota_\alpha^{-1}$, where ι_α is the length function of α. We call a mapping $f : G \to M$ to be a *smooth mapping* , if, for each edge α, the mapping $f_\alpha : [0, \iota(\alpha)] \to M$ is a smooth mapping. We say $f : G \to M$ to be *nondegenerated,* if $\left|\dfrac{\partial f_\alpha}{\partial t}\right| \neq 0$.

Let $f : G \to M$ be a nondegenarate smooth mapping. For each vertex p and each edge α with $p \prec \alpha$, we define $f_{p\alpha} : [0, \iota(\alpha)] \to M$ by $f_{p\alpha}(t) = f \circ \iota_{p\alpha}^{-1}(t)$. For each vertex p of G, we define the *tension vector* $T_f(p)$ at p by

$$T_f(p) = \sum_{\alpha \succ p} \left(1 - \left|\frac{\partial f_{p\alpha}}{\partial t}\right|^{-1}\right) \frac{\partial f_{p\alpha}}{\partial t}(0).$$

In the definition of $T_f(p)$, the term $\left|\frac{\partial f_\alpha}{\partial t}\right|$ is the ratio of expantion of an edge α. If $\left|\frac{\partial f_\alpha}{\partial t}\right| > 1$, the edge α expands. If $\left|\frac{\partial f_\alpha}{\partial t}\right| < 1$ an edge α contracts.

We say a section $V : G \to f^*TM$ to be a *vector field* along to f, if, for each edge α , the mapping $V \circ \iota_\alpha^{-1} : [0, \iota(\alpha)] \to (\iota_\alpha^{-1})^* f^*TM$ is a smooth vector field along to f_α. Put $V_\alpha = V \circ \iota_\alpha^{-1}$. For each vertex p and each edge α with $p \prec \alpha$, put $V_{p\alpha} = V \circ \iota_{p\alpha}^{-1}$. We call $F : (-\varepsilon, \varepsilon) \times G \to M$ a *variation* of $f : G \to M$ if $F(0, x) = f(x)$. Let $F : (-\varepsilon, \varepsilon) \times G \to M$ be a variation of f. For each edge α, we define $F_\alpha : (-\varepsilon, \varepsilon) \times [0, \iota(\alpha)] \to M$ by $F_\alpha(s, t) = F(s, \iota_\alpha^{-1}(t))$. A variation F is *smooth*, if, for each edge α of G, the map $F_\alpha : (-\varepsilon, \varepsilon) \times [0, \iota(\alpha)] \to M$ is smooth. And, for each vertex p and each edge α with $p \prec \alpha$, we define $F_{p\alpha} : (-\varepsilon, \varepsilon) \times [0, \iota(\alpha)] \to M$ by $F_{p\alpha}(s, t) = F(s, \iota_{p\alpha}^{-1}(t))$. A vector field V along to f is the *variation vector field* of F, if the following holds:

$V_\alpha(t) = \frac{\partial F_\alpha}{\partial s}(0, t)$ for $0 \leq t \leq \iota(\alpha)$.

We say a nondegenerated map $f : G \to M$ to be *geodesical* if, for each edge α, the map $f_\alpha : [0, a_\alpha] \to M$ is a geodesic.

Let $f : G \to M$ be geodesical. A vector field $V : G \to f^*TM$ along to f is said to be a *Jacobi field*, if, for each α, V_α is a Jacobi field. We say a Jacobi field V along to f to be the *tension Jacobi field with fixed points* P if

$$V(p) = \begin{cases} 0 & (p \in P) \\ T_f(p) & (p \notin P). \end{cases}$$

We denote by $\tilde{T}_f^P$ the tension Jacobi field along to f with fixed points P. For a smooth variation $F : (-\varepsilon, \varepsilon) \times G \to M$, put $F_s(x) = F(s, x)$. If $F_s : G \to M$ is nondegenerated, we put $T_F(p)(s) = T_{F_s}(p)$ for each vertex p of G. An alteration of our proposal is the following:

$$\frac{\partial}{\partial s}\bigg|_{s=0} \sum_{p \notin P} < T_F(p), T_F(p) > < 0.$$

Then we need to calculate the left side of the above inequality. We have the following:

THEOREM 1.1 *Let G be a graph with length and M a Riemannian manifold. Let $f : G \to M$ be geodesical and F a smooth variation of f. Let $V : G \to f^*TM$ be a Jacobi field along to f with fixed points P. Suppose that V is the variation vector field of F. Then*

$$\frac{\partial}{\partial s}\bigg|_{s=0} \frac{1}{2}\sum_{p\notin P} < T_F(p), T_F(p) >$$

$$= -\sum_{\alpha} \iota(\alpha) \left|\frac{\partial f_\alpha}{\partial t}\right|^{-3} < \nabla_{\frac{\partial}{\partial t}} V_\alpha, \frac{\partial f_\alpha}{\partial t} >< \nabla_{\frac{\partial}{\partial t}} (\tilde{T}_f^P)_\alpha, \frac{\partial f_\alpha}{\partial t} >$$

$$+\sum_{\alpha}\left(1-\left|\frac{\partial f_\alpha}{\partial t}\right|^{-1}\right)\int_0^{\iota(\alpha)} < R(V_\alpha, \frac{\partial f_\alpha}{\partial t})\frac{\partial f_\alpha}{\partial t}, (\tilde{T}_f^P)_\alpha >$$

$$-\sum_{\alpha}\left(1-\left|\frac{\partial f_\alpha}{\partial t}\right|^{-1}\right)\int_0^{\iota(\alpha)} < \nabla_{\frac{\partial}{\partial t}} V_\alpha, \nabla_{\frac{\partial}{\partial t}} (\tilde{T}_f^P)_\alpha > dt.$$

We devote the next section to prove Theorem 1.1. Putting $V = \tilde{T}_f^P$, we immediately have the following:

COROLLARY 1.2 *Let $f : G \to M$ be geodesical and F a smooth variation of f. Suppose that the tension Jacobi field $\tilde{T}_f^P$ with fixed points P is the variation vector field of F.*
Then

$$\frac{\partial}{\partial s}\bigg|_{s=0} \frac{1}{2}\sum_{p\notin P} < T_F(p), T_F(p) >$$

$$= -\sum_{\alpha} \iota(\alpha) \left|\frac{\partial f_\alpha}{\partial t}\right|^{-3} < \nabla_{\frac{\partial}{\partial t}} (\tilde{T}_f^P)_\alpha, \frac{\partial f_\alpha}{\partial t} >^2$$

$$+\sum_{\alpha}\left(1-\left|\frac{\partial f_\alpha}{\partial t}\right|^{-1}\right)\int_0^{\iota(\alpha)} < R((\tilde{T}_f^P)_\alpha, \frac{\partial f_\alpha}{\partial t})\frac{\partial f_\alpha}{\partial t}, (\tilde{T}_f^P)_\alpha >$$

$$-\sum_{\alpha}\left(1-\left|\frac{\partial f_\alpha}{\partial t}\right|^{-1}\right)\int_0^{\iota(\alpha)} < \nabla_{\frac{\partial}{\partial t}} (\tilde{T}_f^P)_\alpha, \nabla_{\frac{\partial}{\partial t}} (\tilde{T}_f^P)_\alpha > dt.$$

By Corollary 1.2, we have the following:

COROLLARY 1.3 *Let $f : G \to M$ be geodesical and F a variation of f. Suppose that the tension Jacobi field $\tilde{T}_f^P$ with fixed points P is the variation vector field of F. Suppose that M has a negative curvature and $\left|\frac{\partial f_\alpha}{\partial t}\right| > 1$ for each edge α and that there exists a vertex $p \notin P$ such that $T_f(p) \neq 0$.*
Then

$$\frac{\partial}{\partial s}\bigg|_{s=0} \sum_{p\notin P} < T_F(p), T_F(p) > < 0.$$

PROOF Suppose that G is connected, because it is enough to give the proof for each connected component. Suppose that there exists an edge α such that $\nabla_{\frac{\partial}{\partial t}}(\tilde{T}_f^P)_\alpha \neq 0$. Then $< \nabla_{\frac{\partial}{\partial t}}(\tilde{T}_f^P)_\alpha, \nabla_{\frac{\partial}{\partial t}}(\tilde{T}_f^P)_\alpha > > 0$. By Lemma 1.2, immediately we have

$$\frac{\partial}{\partial s}\bigg|_{s=0} \frac{1}{2} \sum_{p \in P} < T_F(p), T_F(p) > < 0.$$

Now suppose that $\nabla_{\frac{\partial}{\partial t}}(\tilde{T}_f^P)_\alpha = 0$ for each edge α. If $P \neq \emptyset$, than $(\tilde{T}_f^P)_\alpha = 0$ for each vertex p. Then $P = \emptyset$, since there exists a vertex p such that $T_f(p) \neq 0$. Suppose that $T_f(p)$ has the same direction to one of $\frac{\partial f_{p\alpha}}{\partial t}(0)$ and $-\frac{\partial f_{p\alpha}}{\partial t}(0)$, for each $p \prec \alpha$. Then, for p and α with $p \prec \alpha$, there exists a unique vector field $V_{p\alpha}$ on G, such that

$$V_{p\alpha}(p) = \left(1 - \left|\frac{\partial f_{p\alpha}}{\partial t}\right|^{-1}\right) \frac{\partial f_{p\alpha}}{\partial t}(0)$$

and $\nabla_{\frac{\partial}{\partial t}}(V_{p\alpha})_\beta = 0$ for each edge β, where $(V_{p\alpha})_\beta = V_{p\alpha} \circ \iota_\beta^{-1}$. Note that $\tilde{T}_f^P = \sum_{\alpha \succ p} V_{p\alpha}$, for each p. Then $\# V(G)\tilde{T}_f^P = \sum_p \sum_{\alpha \succ p} V_{p\alpha}$ where $V(G)$ is the sets of vertices of G. Since $V_{p\alpha} + V_{q\alpha} = 0$ for $p, q \prec \alpha$, it follows that $\sum_p \sum_{\alpha \succ p} V_{p\alpha} = \sum_\alpha \sum_{p \prec \alpha} V_{p\alpha} = 0$. Then $\tilde{T}_f^P = 0$. This is contradiction. Then there exist a vertex p and an edge α such that $p \prec \alpha$ and that $T_f(p)$ has a different direction from each of $\frac{\partial f_{p\alpha}}{\partial t}(0)$ and $-\frac{\partial f_{p\alpha}}{\partial t}(0)$. Then $< R((\tilde{T}_f^P)_\alpha, \frac{\partial f_\alpha}{\partial t})\frac{\partial f_\alpha}{\partial t}, (\tilde{T}_f^P)_\alpha > < 0$. By Lemma 1.2, we have

$$\frac{\partial}{\partial s}\bigg|_{s=0} \frac{1}{2} \sum_{p \notin P} < T_F(p), T_F(p) > < 0.$$

q.e.d.

COROLLARY 1.4 *Let $f : G \to \boldsymbol{R}^n$ be geodesical and F a variation of f. Suppose that Suppose that the tension Jacobi field $\tilde{T}_f^P$ with fixed points P is the variation vector field of F.Suppose that $\left|\frac{\partial f_\alpha}{\partial t}\right| > 1$ for each edge α and that there exists a vertex $p \notin P$ such that $T_f(p) \neq 0$.Then*

$$\frac{\partial}{\partial s}\bigg|_{s=0} \sum_{p \notin P} < T_F(p), T_F(p) > < 0.$$

PROOF By Corollary 1.2, immediately we have

$$\frac{\partial}{\partial s}\bigg|_{s=0} \frac{1}{2} \sum_{p \notin P} < T_F(p), T_F(p) > \leq 0.$$

We will prove that $\left.\frac{\partial}{\partial s}\right|_{s=0} \sum_{p\notin P} < T_F(p), T_F(p) >= 0$ implies $T_f(p) = 0$ for $p \notin P$. Consequently, we have $\left.\frac{\partial}{\partial s}\right|_{s=0} \frac{1}{2}\sum_{p\notin P} < T_F(p), T_F(p) >< 0$. Suppose that $\left.\frac{\partial}{\partial s}\right|_{s=0} \sum_{p\notin P} < T_F(p), T_F(p) > = 0$. Note that $< R((\tilde{T}_f^P)_\alpha, \frac{\partial f_\alpha}{\partial t})\frac{\partial f_\alpha}{\partial t}, (\tilde{T}_f^P)_\alpha >=$ 0. By Corollary 1.2, we have $\nabla_{\frac{\partial}{\partial t}}(\tilde{T}_f^P)_\alpha = 0$. If $P \neq \emptyset$, then $\tilde{T}_f^P(p) = 0$ for each vertex p. Then $T_f(p) = 0$ for each vertex $p \notin P$. Next suppose that $P = \emptyset$. For any tangent vector V in $\boldsymbol{R}^n$, we denote by V_0 the vector which is parallelly translated V to the origin of $\boldsymbol{R}^n$. Then V_0 is determined independently on the choice of pathes. Not that

$$\sum_{p\prec\alpha}\left(\left(1-\left|\frac{\partial f_{p\alpha}}{\partial t}\right|^{-1}\right)\frac{\partial f_{p\alpha}}{\partial t}(0)\right)_0 = 0,$$

for each edge α, and

$$\sum_p (T_f(p))_0 = \sum_p \sum_{\alpha\succ p}\left(\left(1-\left|\frac{\partial f_{p\alpha}}{\partial t}\right|^{-1}\right)\frac{\partial f_{p\alpha}}{\partial t}(0)\right)_0 .$$

Then $\sum_p (T_f(p))_0 = 0$. Since $T_f(p)$ are parallel to each other, we have $T_f(p) = 0$.

q.e.d.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 *Let $f : [a,b] \to M$ be a geodesic. Let Y be a Jacobi field along to f. Then*

$$< Y, \frac{\partial f}{\partial t} > (t) =< Y(a), \frac{\partial f}{\partial t}(a) > \frac{b-t}{b-a} + < Y(b), \frac{\partial f}{\partial t}(b) > \frac{t-a}{b-a}.$$

Proof Note that $\nabla_{\frac{\partial}{\partial t}}\frac{\partial f}{\partial t} = 0$ and $\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}} Y = -R(Y, \frac{\partial f}{\partial t})\frac{\partial f}{\partial t}$. Then

$$\begin{aligned}\frac{\partial^2}{\partial t^2} < Y, \frac{\partial f}{\partial t} > &= - < \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}} Y, \frac{\partial f}{\partial t} > \\ &=< R(Y, \frac{\partial f}{\partial t})\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} > \\ &= 0.\end{aligned}$$

Then we have $< Y, \frac{\partial f}{\partial t} > (t)$ is a linear function. Then

$$< Y, \frac{\partial f}{\partial t} > (t) =< Y(a), \frac{\partial f}{\partial t}(a) > \frac{b-t}{b-a} + < Y(b), \frac{\partial f}{\partial t}(b) > \frac{t-a}{b-a}.$$

q.e.d.

LEMMA 2.2 *Let $f : [a,b] \to M$ be a geodesic. Let Y be a Jacobi field along to f and let V be a vector field along to f. Then*

$$\begin{aligned} < \nabla_{\frac{\partial}{\partial t}} Y(b), V(b) > - < \nabla_{\frac{\partial}{\partial t}} Y(a), V(a) > = &- \int_a^b < R(Y, \frac{\partial f}{\partial t}) \frac{\partial f}{\partial t}, V > dt \\ &+ \int_a^b < \nabla_{\frac{\partial}{\partial t}} Y, \nabla_{\frac{\partial}{\partial t}} V > dt. \end{aligned}$$

PROOF We have $\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} Y = -R(Y, \frac{\partial f}{\partial t}) \frac{\partial f}{\partial t}$ and

$$\int_a^b \frac{\partial}{\partial t} < \nabla_{\frac{\partial}{\partial t}} Y, V > dt = \int_a^b < \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} Y, V > dt + \int_a^b < \nabla_{\frac{\partial}{\partial t}} Y, \nabla_{\frac{\partial}{\partial t}} V > dt.$$

Then we have the result.

PROOF OF THEOREM 1.1 Note that

$$\nabla_{\frac{\partial}{\partial s}} \Big|_{s=0} \frac{\partial F_{p\alpha}}{\partial t} = \nabla_{\frac{\partial}{\partial t}} \Big|_{s=0} \frac{\partial F_{p\alpha}}{\partial s} = \nabla_{\frac{\partial}{\partial t}} V_{p\alpha}.$$

For each vertex p of G, we have

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}} \Big|_{s=0} T_F(p) = &\sum_{\alpha \succ p} \left| \frac{\partial f_{p\alpha}}{\partial t} \right|^{-3} (0) < \nabla_{\frac{\partial}{\partial t}} \Big|_{t=0} V_{p\alpha}, \frac{\partial f_{p\alpha}}{\partial t}(0) > \frac{\partial f_{p\alpha}}{\partial t}(0) \\ &+ \sum_{\alpha \succ p} \left(1 - \left| \frac{\partial f_{p\alpha}}{\partial t} \right|^{-1} (0) \right) \nabla_{\frac{\partial}{\partial t}} \Big|_{t=0} V_{p\alpha}. \end{aligned}$$

Note that the following facts:

1. $\left| \frac{\partial f_{p\alpha}}{\partial t} \right| = \left| \frac{\partial f_{\alpha}}{\partial t} \right|$ is constant, since $f_{p\alpha}$ is geodesic.
2. $< \nabla_{\frac{\partial}{\partial t}} V_{p\alpha}, \frac{\partial f_{p\alpha}}{\partial t} > =< \nabla_{\frac{\partial}{\partial t}} V_{\alpha}, \frac{\partial f_{\alpha}}{\partial t} >$ is constant by Lemma 2.1,
3. $\tilde{T}_f^P(p) = 0$ for $p \in P$.

Then

$$\frac{\partial}{\partial s}\bigg|_{s=0} \frac{1}{2}\sum_{p\notin P} < T_F(p), T_F(p) >$$

$$= \sum_{p\notin P} < \nabla_{\frac{\partial}{\partial s}}\Big|_{s=0} T_F(p), T_f(p) >$$

$$= \sum_{p} < \nabla_{\frac{\partial}{\partial s}}\Big|_{s=0} T_F(p), \tilde{T}_f^P(p) >$$

$$= \sum_{\alpha} \left|\frac{\partial f_\alpha}{\partial t}\right|^{-3} < \nabla_{\frac{\partial}{\partial t}} V_\alpha, \frac{\partial f_\alpha}{\partial t} > \sum_{p\prec\alpha} < \frac{\partial f_{p\alpha}}{\partial t}(0), \tilde{T}_f^P(p) >$$

$$+ \sum_{\alpha}\left(1 - \left|\frac{\partial f_\alpha}{\partial t}\right|^{-1}\right) \sum_{p\prec\alpha} < \nabla_{\frac{\partial}{\partial t}}\Big|_{t=0} V_{p\alpha}, \tilde{T}_f^P(p) > .$$

Since $\frac{\partial}{\partial t} < \frac{\partial f_{p\alpha}}{\partial t}, \tilde{T}_f^P(p) >$ is constant by Lemma 2.1, for each α, it follows that

$$\sum_{p\prec\alpha} < \frac{\partial f_{p\alpha}}{\partial t}(0), \tilde{T}_f^P(p) > = -\int_0^{\iota(\alpha)} \frac{\partial}{\partial t} < \frac{\partial f_\alpha}{\partial t}, (\tilde{T}_f^P)_\alpha > dt$$

$$= -\iota(\alpha) < \frac{\partial f_\alpha}{\partial t}, \nabla_{\frac{\partial}{\partial t}}(\tilde{T}_f^P)_\alpha > .$$

By Lemma 2.2, for each edge α, we have

$$\sum_{p\prec\alpha} < \nabla_{\frac{\partial}{\partial t}}|_{t=0} V_{p\alpha}, \tilde{T}_f^P(p) > = \int_0^{\iota(\alpha)} < R(V_\alpha, \frac{\partial f_\alpha}{\partial t})\frac{\partial f_\alpha}{\partial t}, (\tilde{T}_f^P)_\alpha > dt$$

$$- \int_0^{\iota(\alpha)} < \nabla_{\frac{\partial}{\partial t}} V_\alpha, \nabla_{\frac{\partial}{\partial t}}(\tilde{T}_f^P)_\alpha > dt.$$

Then we have

$$\frac{\partial}{\partial s}\bigg|_{s=0} \frac{1}{2}\sum_{p\notin P} < T_F(p), T_F(p) >$$

$$= -\sum_{\alpha} \iota(\alpha) \left|\frac{\partial f_\alpha}{\partial t}\right|^{-3} < \nabla_{\frac{\partial}{\partial t}} V_\alpha, \frac{\partial f_\alpha}{\partial t} > \cdot < \nabla_{\frac{\partial}{\partial t}}(\tilde{T}_f^P)_\alpha, \frac{\partial f_\alpha}{\partial t} >$$

$$+ \sum_{\alpha}(1 - \left|\frac{\partial f_\alpha}{\partial t}\right|^{-1}) \int_0^{\iota(\alpha)} < R(V_\alpha, \frac{\partial f_\alpha}{\partial t})\frac{\partial f_\alpha}{\partial t}, (\tilde{T}_f^P)_\alpha > dt$$

$$- \sum_{\alpha}(1 - \left|\frac{\partial f_\alpha}{\partial t}\right|^{-1}) \int_0^{\iota(\alpha)} < \nabla_{\frac{\partial}{\partial t}} V_\alpha, \nabla_{\frac{\partial}{\partial t}}(\tilde{T}_f^P)_\alpha > dt.$$

q.e.d.

3 Examples

In this section, we construct examples such that $\left.\frac{\partial}{\partial s}\right|_{s=0} < T_F(p), T_F(p) > > 0$.

Let $K_{1,3}$ be the graph defined by $K_{1,3} = p*\{\alpha, \beta, \gamma\}$. Here p and $\{\alpha, \beta, \gamma\}$ are one point and three points, respectively. Let $\iota_\alpha : p*\alpha \to [0, L]$, $\iota_\beta : p*\beta \to [0, l]$, and $\iota_\gamma : p*\gamma \to [0, l]$, be the length functions such that $\iota_\alpha(p) = \iota_\beta(p) = \iota_\gamma(p) = 0$, $\iota_\alpha(\alpha) = L$, $\iota_\beta(\beta) = l$ and $\iota_\gamma(\gamma) = l$. Throughout this section, we use these notations.

3.1 The case of $\boldsymbol{R}^2$

We define $f : K_{1,3} \to \boldsymbol{R}^2$ as follows:

1. $f_{p*\alpha} = f \circ \iota_\alpha^{-1} : [0, L] \to \boldsymbol{R}^2$ *defined by* $f_{p*\alpha}(t) = \left(\frac{t}{m}, 0\right)$.
2. $f_{p*\beta} = f \circ \iota_\beta^{-1} : [0, l] \to \boldsymbol{R}^2$ *defined by* $f_{p*\beta}(t) = \left(0, \frac{t}{n}\right)$.
3. $f_{p*\gamma} = f \circ \iota_\gamma^{-1} : [0, l] \to \boldsymbol{R}^2$ *defined by* $f_{p*\gamma}(t) = \left(0, -\frac{t}{n}\right)$.

Then we have the following:

Proposition 3.1 *Let $f : K_{1,3} \to \boldsymbol{R}^2$ be as aboves. and F a variation of f. Suppose that the tension Jacobi field $\tilde{T}_f^P$ along to f with fixed points $P = \{\alpha, \beta, \gamma\}$ is the variation vector fieldof F. If $m \neq 1$, and $\frac{2(n-1)}{l} > \frac{1}{L}$ then $\left.\frac{\partial}{\partial s}\right|_{s=0} < T_F(p), T_F(p) > > 0$.*

Proof Let (x, y) be the coordinate of $\boldsymbol{R}^2$. Then we have $\frac{\partial f_{p*\alpha}}{\partial t}(0) = \frac{1}{m}\left.\frac{\partial}{\partial x}\right|_{(0,0)}$, $\frac{\partial f_{p*\beta}}{\partial t}(0) = \frac{1}{n}\left.\frac{\partial}{\partial y}\right|_{(0,0)}$, $\frac{\partial f_{p*\gamma}}{\partial t}(0) = -\frac{1}{n}\left.\frac{\partial}{\partial y}\right|_{(0,0)}$ and we have $T_f(p) = \frac{1-m}{m}\left.\frac{\partial}{\partial x}\right|_{(0,0)}$. Here $T_f(p)$ is the tension vector at p. Then, for the tension Jacobi field $\tilde{T}_f^P$, we have the following:

$$
\begin{aligned}
\left(\tilde{T}_f^P\right)_{p*\alpha}(t) &= \frac{1-m}{m}\frac{L-t}{L}\left.\frac{\partial}{\partial x}\right|_{(\frac{t}{m},0)} \\
\left(\tilde{T}_f^P\right)_{p*\beta}(t) &= \frac{1-m}{m}\frac{l-t}{l}\left.\frac{\partial}{\partial x}\right|_{(0,\frac{t}{n})} \\
\left(\tilde{T}_f^P\right)_{p*\gamma}(t) &= \frac{1-m}{m}\frac{l-t}{l}\left.\frac{\partial}{\partial x}\right|_{(0,-\frac{t}{n})}.
\end{aligned}
$$

Then

$$\left(\nabla_{\frac{\partial}{\partial t}}\tilde{T}_f^P\right)_{p*\alpha}(t) = \frac{m-1}{mL}\frac{\partial}{\partial x}\Big|_{(\frac{t}{m},0)}$$
$$\left(\nabla_{\frac{\partial}{\partial t}}\tilde{T}_f^P\right)_{p*\beta}(t) = \frac{m-1}{ml}\frac{\partial}{\partial x}\Big|_{(0,\frac{t}{n})}$$
$$\left(\nabla_{\frac{\partial}{\partial t}}\tilde{T}_f^P\right)_{p*\gamma}(t) = \frac{m-1}{ml}\frac{\partial}{\partial x}\Big|_{(0,-\frac{t}{n})}.$$

By Corollary 1.2, for a variation F tangent to the tension Jacobi field $\tilde{T}_f^P$, we have

$$\begin{aligned}
&\frac{\partial}{\partial s}\Big|_{s=0}\frac{1}{2} < T_F(p), T_F(p) > \\
&= - Lm^3 < \frac{m-1}{mL}\frac{\partial}{\partial x}, \frac{1}{m}\frac{\partial}{\partial x} >^2 \\
&\quad - ln^3 < \frac{m-1}{ml}\frac{\partial}{\partial x}, \frac{1}{n}\frac{\partial}{\partial x} >^2 \\
&\quad - ln^3 < \frac{m-1}{ml}\frac{\partial}{\partial x}, -\frac{1}{n}\frac{\partial}{\partial x} >^2 \\
&\quad - (1-m)\int_0^L < \frac{m-1}{mL}\frac{\partial}{\partial x}\Big|_{(\frac{t}{m},0)}, \frac{m-1}{mL}\frac{\partial}{\partial x}\Big|_{(\frac{t}{m},0)} > dt \\
&\quad - (1-n)\int_0^l < \frac{m-1}{ml}\frac{\partial}{\partial x}\Big|_{(0,\frac{t}{n})}, \frac{m-1}{ml}\frac{\partial}{\partial x}\Big|_{(0,\frac{t}{n})} > dt \\
&\quad - (1-n)\int_0^l < \frac{m-1}{ml}\frac{\partial}{\partial x}\Big|_{(0,-\frac{t}{n})}, \frac{m-1}{ml}\frac{\partial}{\partial x}\Big|_{(0,-\frac{t}{n})} > dt.
\end{aligned}$$

Then we have $\frac{\partial}{\partial s}\Big|_{s=0}\frac{1}{2} < T_F(p), T_F(p) >= \left(1-\frac{1}{m}\right)^2\left(\frac{2(n-1)}{l} - \frac{1}{L}\right)$.
If $m \neq 1$ and $\frac{2(n-1)}{l} > \frac{1}{L}$ then $\frac{\partial}{\partial s}\Big|_{s=0} < T_F(p), T_F(p) > > 0$.

q.e.d

3.2 The case of S^2

We define $f : K_{1,3} \to S^2$ as follows:

$f_{p*\alpha} = f \circ \iota_\alpha^{-1} : [0, L] \to S^2$ defined by $f_{p*\alpha}(t) = (1,0,0)\cos\frac{W}{L}t + (0,-1,0)\sin\frac{W}{L}t.$
$f_{p*\beta} = f \circ \iota_\beta^{-1} : [0, l] \to S^2$ defined by $f_{p*\beta}(t) = (1,0,0)\cos\frac{w}{l}t + (0,0,1)\sin\frac{w}{l}t.$
$f_{p*\gamma} = f \circ \iota_\gamma^{-1} : [0, l] \to S^2$ defined by $f_{p*\gamma}(t) = (1,0,0)\cos\frac{w}{l}t + (0,0,-1)\sin\frac{w}{l}t.$

Here L, l, W and w are positive numbers. Then we have the following:

PROPOSITION 3.2 *Let $f : K_{1,3} \to S^2$ be as aboves and F a variation of f.Suppose that the tension Jacobi field $\tilde{T}_f$ along to f with fixed points $P = \{\alpha, \beta, \gamma\}$ is the variation vector field F. If $\sin w \neq 0$, then*

$$\frac{\partial}{\partial s}\bigg|_{s=0} \frac{1}{2} < T_F(p), T_F(p) >= \left(\frac{W}{L} - 1\right)^2 \left(-\frac{1}{L} + 2\left(\frac{w}{l} - 1\right)\frac{-\cos w}{\sin w}\right).$$

Immediately, we have the following. Then we omit the proof.

COROLLARY 3.3 *Suppose that $W \neq L, w > l$ and $\frac{\pi}{2} < w < \pi$.*
If $L > \left(2\left(\frac{w}{l} - 1\right)\frac{-\cos w}{\sin w}\right)^{-1}$ then $\frac{\partial}{\partial s}\Big|_0 < T_F(p), T_F(p) > > 0$.

PROOF OF PROPOSITION 3.2
Since $\frac{\partial f_{p*\alpha}}{\partial t}(0) = -\frac{W}{L}\frac{\partial}{\partial y}\Big|_{(1,0,0)}$, $\frac{\partial f_{p*\beta}}{\partial t}(0) = \frac{w}{l}\frac{\partial}{\partial z}\Big|_{(1,0,0)}$,
and $\frac{\partial f_{p*\gamma}}{\partial t}(0) = -\frac{w}{l}\frac{\partial}{\partial z}\Big|_{(1,0,0)}$, it follows that $T_f(p) = \left(-\frac{W}{L} + 1\right)\frac{\partial}{\partial y}\Big|_{(1,0,0)}$.
Put $\tau = \frac{W}{L} - 1$. For some $\varepsilon > 0$, we can define $\tilde{w} : (-\varepsilon, \varepsilon) \to \boldsymbol{R}$ such that $\tilde{w}(0) = w$, and $\cos\tilde{w} = \cos s\tau \cos w$.Then we have $\frac{\partial \tilde{w}}{\partial s}\Big|_{s=0} = 0$.
We define a variation $F : (-\varepsilon, \varepsilon) \times K_{1,3} \to S^2$ of f as follows:

$$\begin{aligned}
F_{p*\alpha}(s,t) =& (\cos s\tau, -\sin s\tau, 0)\cos\frac{W - s\tau}{L}t \\
&+ (-\sin s\tau, -\cos s\tau, 0)\sin\frac{W - s\tau}{L}t \\
F_{p*\beta}(s,t) =& (\cos s\tau, -\sin s\tau, 0)\cos\frac{\tilde{w}}{l}t \\
&+ \left(\frac{\sin^2 s\tau\cos w}{\sin\tilde{w}}, \frac{\sin s\tau\cos s\tau\cos w}{\sin\tilde{w}}, \frac{\sin w}{\sin\tilde{w}}\right)\sin\frac{\tilde{w}}{l}t \\
F_{p*\gamma}(s,t) =& (\cos s\tau, -\sin s\tau, 0)\cos\frac{\tilde{w}}{l}t \\
&+ \left(\frac{\sin^2 s\tau\cos w}{\sin\tilde{w}}, \frac{\sin s\tau\cos s\tau\cos w}{\sin\tilde{w}}, -\frac{\sin w}{\sin\tilde{w}}\right)\sin\frac{\tilde{w}}{l}t.
\end{aligned}$$

Then $F_{p*\alpha}(0,t) = f_{p*\alpha}(t)$, $F_{p*\beta}(0,t) = f_{p*\beta}(t)$, and $F_{p*\gamma}(0,t) = f_{p*\gamma}(t)$, hence F is a variation of f.
Putting $F_s(x) = F(s,x)$, we have F_s is a geodesical. Note that

$$\begin{aligned}
F_{p*\alpha}(s,L) &= f_{p*\alpha}(L) = f(\alpha) \\
F_{p*\beta}(s,l) &= f_{p*\beta}(l) = f(\beta) \\
F_{p*\gamma}(s,l) &= f_{p*\gamma}(l) = f(\gamma).
\end{aligned}$$

Since
$$\frac{\partial F_{p*\alpha}}{\partial s}(0,0) = \frac{\partial F_{p*\beta}}{\partial s}(0,0) = \frac{\partial F_{p*\gamma}}{\partial s}(0,0) = -\tau \frac{\partial}{\partial y} = T_f(p),$$

it follow that tension Jacobi field $\tilde{T}_f^{\{\alpha,\beta,\gamma\}}$ is the variation vector field of F and has fixed points $\{\alpha, \beta, \gamma\}$. Moreover

$$\begin{aligned}
\left.\frac{\partial F_{p*\alpha}}{\partial t}\right|_{t=0} &= \frac{W - s\tau}{L}\left(-\sin s\tau \frac{\partial}{\partial x} - \cos s\tau \frac{\partial}{\partial y}\right) \\
\left.\frac{\partial F_{p*\gamma}}{\partial t}\right|_{t=0} &= \frac{\tilde{w}}{l}\left(\frac{\sin^2 s\tau \cos w}{\sin \tilde{w}}\frac{\partial}{\partial x} + \frac{\sin s\tau \cos s\tau \cos w}{\sin \tilde{w}}\frac{\partial}{\partial y} + \frac{\sin w}{\sin \tilde{w}}\frac{\partial}{\partial z}\right) \\
\left.\frac{\partial F_{p*\gamma}}{\partial t}\right|_{t=0} &= \frac{\tilde{w}}{l}\left(\frac{\sin^2 s\tau \cos w}{\sin \tilde{w}}\frac{\partial}{\partial x} + \frac{\sin s\tau \cos s\tau \cos w}{\sin \tilde{w}}\frac{\partial}{\partial y} - \frac{\sin w}{\sin \tilde{w}}\frac{\partial}{\partial z}\right)
\end{aligned}$$

Note that $\left|\frac{\partial F_{p*\alpha}}{\partial t}|_{t=0}\right| = \left|\frac{W - s\tau}{L}\right|$, and $\left|\frac{\partial F_{p*\beta}}{\partial t}|_{t=0}\right| = \left|\frac{\tilde{w}}{l}\right|$, $\left|\frac{\partial F_{p*\gamma}}{\partial t}|_{t=0}\right| = \left|\frac{\tilde{w}}{l}\right|$.
If ε is sufficiently small, then, for $s \in (-\varepsilon, \varepsilon)$, we have

$$T_F(p) = \left\{\left(\frac{W - s\tau}{L} - 1\right) - 2\left(\frac{\tilde{w}}{l} - 1\right)\frac{\sin s\tau \cos w}{\sin \tilde{w}}\right\}\left(-\sin s\tau \frac{\partial}{\partial x} - \cos s\tau \frac{\partial}{\partial y}\right).$$

Note that $\left.\frac{\partial \tilde{w}}{\partial s}\right|_{s=0} = 0$. Then we have

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \frac{1}{2} < T_F(p), T_F(p) >= \left(\frac{W}{L} - 1\right)^2\left(-\frac{1}{L} + 2\left(\frac{w}{l} - 1\right)\frac{-\cos w}{\sin w}\right).$$

q.e.d.

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Distributions on Graphes and Walks

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1 Introduction and the statement of results

We generalize the notion of Hamiltonian graphs. Let G be a graph. Let $V(G)$ and $E(G)$ be the set of all vertices of G and that of all edges of G,respectively(c.f. [1], [2]). We call a mapping $D : V(G) \to Z/nZ$ a mod n distribution on G. A mapping $\sigma : \{0,1,2,\ldots,m\} \to V(G)$ is said to be a walk with length m or merely to be a walk if $<\sigma(i), \sigma(i+1)>$ are edges in $E(G)$ for $i = 0,1,\ldots,m-1$. A walk σ is said to be closed if $\sigma(0) = \sigma(m)$, where m is the length of σ.

Let $\sigma : \{0,1,2,\ldots,m\} \to V(G)$ be a walk on G. We define a distribution $D(\sigma) : V(G) \to Z/nZ$ on G by $D(\sigma)(v) \equiv \#\{\sigma^{-1}(v) | \sigma^{-1}(v) \leq m-1\}$ mod n. We take this definition of $D(\sigma)$, in order to avoid that $0, m \in \sigma^{-1}\sigma(0)$ for a closed walk σ. We call $D(\sigma)$ the mod n distribution induced by σ. A graph G is Hamiltonian if there exists a walk σ such that $D(\sigma)(v) = 1$ in Z, for all v in $V(G)$. In this paper, we study mod n distribution for $n > 1$. Except for bipartite graphs, we have the following:

Theorem 1 *Let G be a connected graph and D a* mod n *distribution on G. Let u_0 and u_1 be vertices of G. If $n > 1$ and G is not a bipartite graph, then there exists a walk σ on G such that $\sigma(0) = u_0, \sigma(m) = u_1$ and $D(\sigma) = D$, where m is the length of σ.*

Let G be a connected bipartite graph, that is, the set $V(G)$ of vertices of G is divided into two part $V_0(G)$ and $V_1(G)$, where vertices in same part are not adjacent to each other. Let D be a mod n distribution on a connected bipartite graph G. We set $S_k(D) = \sum_{v \in V_k(G)} D(v)$, for $k = 0,1$, and $S(D) = S_0(D) - S_1(D)$. Throughout this paper we use these notations. For bipartite graphs, we have the following:

Theorem 2 *Let D be a* mod n *distribution on a connected bipartite graph G and $n > 1$. Let u_0 and u_1 be vertices of G. If u_0 and u_1 are in the same part $V_k(G)$ and $S(D) \equiv 0 \mod n$, then there exists a walk σ such that $\sigma(0) = u_0, \sigma(m) = u_1$, and $D(\sigma) = D$, where m is the length of σ. If u_k is in $V_k(G)$ for each $k = 0,1$, and $S(D) \equiv 1 \mod n$, then there exists a walk σ such that $\sigma(0) = u_0, \sigma(m) = u_1$ and $D(\sigma) = D$, where m is the length of σ.*

2 Proof of Theorems

To prove Theorems, we need the following:

Lemma 3 *Let G be not a bipartite graph, and $n > 1$. Then, for any vertex u in $V(G)$, there exists a closed walk τ such that $\tau(0) = u$ and*

$$D(\tau)(v) \equiv \begin{cases} 1 \mod n & \text{for } v = u \\ 0 \mod n & \text{for } v \neq u. \end{cases}$$

Lemma 4 *Let G be a connected graph and D a $\bmod n$ distribution. If $n > 1$, then, for any $u_0, u_1 \in V(G)$, there exists a walk τ such that $\tau(0) = u_0, \tau(m) = u_1$ and $D(\tau)(v) = D(v)$ $(v \neq u_1)$, where m is the length of τ.*

We devote next section to prove Lemma 3 and Lemma 4. We introduce the notation of connection of walks. Let τ_1 and τ_2 be walks with length m_1 and m_2, respectively. Suppose that $\tau_1(m_1) = \tau_2(0)$. We define a $\bmod n$ distribution $\tau_1 \# \tau_2$ by

$$\tau_1 \# \tau_2(k) = \begin{cases} \tau_1(k) & 0 \leq k \leq m_1 \\ \tau_2(k - m_1) & m_1 \leq k \leq m_1 + m_2. \end{cases}$$

For a closed walk τ and a positive integer p, we define $p\tau$ by $p\tau = ((p-1)\tau)\#\tau$. By the definition, we immediately have the following. Then we omit the proof.

Lemma 5 *For $v \in V(G)$, the following holds*

$$D(\tau_1 \# \tau_2)(v) \equiv D(\tau_1)(v) + D(\tau_2)(v) \mod n.$$

Proof of Theorem 1

By Lemma 4, we have a walk τ_1 such that $\tau_1(0) = u_0, \tau_1(m) = u_1$ and $D(\tau_1)(v) = D(v)$ for $v \neq u_1$, where m is the length of τ_1. By Lemma 3, we have a closed walk τ_2 such that $\tau_2(0) = u_1$, and

$$D(\tau_2)(v) \equiv \begin{cases} 1 \mod n & \text{for } v = u_1 \\ 0 \mod n & \text{for } v \neq u_1. \end{cases}$$

We put $\sigma = \tau_1 \# (D(u_1) - D(\tau_1)(u_1))\tau_2$. By Lemmma 5, we have $D(\sigma)(v) \equiv D(\tau_1)(v) + (D(u_1) - D(\tau_1)(u_1))D(\tau_2)(v) \mod n$. If $v \neq u_1$, then $D(\sigma)(v) \equiv D(\tau_1)(v) \equiv D(v) \mod n$. If $v = u_1$, then $D(\sigma)(v) \equiv D(\tau_1)(u_1) + (D(u_1) - D(\tau_1)(u_1))D(\tau_2)(u_1) \equiv D(u_1) \mod n$. Then $D(\sigma) = D$.

q.e.d.

Note that $S_k((D)(\sigma)) \equiv \#\{i \in Z | 0 \leq i \leq m - 1, \sigma(i) \in V_k(G)\} \mod n$. Immediately we have the following. The we omit the proof.

Lemma 6 *Let σ be a walk with m length on a connected bipartite graph G. If m is even, then $S(D(\sigma)) \equiv 0 \pmod n$. If m is odd and $\sigma(0)$ is in $V_0(G)$, then $S(D(\sigma)) \equiv 1 \pmod n$.*

Proof of Theorem 2

By Lemma 4, there exists a walk σ such that $\sigma(0) = u_0$, and $\sigma(m) = u_1$, where m is the length of σ and that $D(\sigma)(v) \equiv D(v) \mod n$ for $v \neq u_1$.

Suppose that u_0 and u_1 are in the same part $V_k(G)$ for $k = 0$ or 1. Then m is even. Moreover suppose that $S(D) \equiv 0 \mod n$. By Lemma 6, we have $S(D(\sigma)) \equiv S(D) \mod n$. Then $D(\sigma)(u_1) \equiv D(u_1) \mod n$. Then $D(\sigma) = D$.

Suppose that u_k is in $V_k(G)$ for $k = 0, 1$. Then m is odd. Moreover suppose that $S(D) \equiv 1 \mod n$. By Lemma 6, we have $S(D(\sigma)) \equiv S(D) \mod n$. Then $D(\sigma)(u_1) \equiv D(u_1) \mod n$. Then $D(\sigma) = D$.

q.e.d.

3 Proof of Lemmas

Proof of Lemma 3

Note that G is not a bipartite graph. Then we can choose a closed walk ρ such that $\rho(0) = u$ and its length is odd. Let $2p+1$ be the length of ρ. For $k = 0, 1, \ldots, p$, define walks τ^{2k} with length 1 by $\tau^{2k}(0) = \rho(2k)$ and $\tau^{2k}(1) = \rho(2k+1)$. Then

$$D(\tau^{2k})(v) \equiv \begin{cases} 1 \mod n & \text{for } v = \rho(2k) \\ 0 \mod n & \text{for } v \neq \rho(2k) \end{cases}$$

For $k = 1, 2, \ldots, p$, define walks τ^{2k-1} with length $2n-1$ by $\tau^{2k-1}(2i) = \rho(2k-1)$ and $\tau^{2k-1}(2i+1) = \rho(2k)$ $(i = 0, 1, \ldots, n-1)$. Then

$$D(\tau^{2k-1})(v) \equiv \begin{cases} n-1 \mod n & \text{for } v = \rho(2k) \\ 0 \mod n & \text{for } v \neq \rho(2k). \end{cases}$$

Set $\tau = \tau^0 \# \tau^1 \# \cdots \# \tau^{2p+1}$.
Then

$$D(\tau)(v) \equiv \begin{cases} 1 \mod n & \text{for } v = \rho(0) \\ 0 \mod n & \text{for } v \neq \rho(0). \end{cases}$$

q.e.d.

Lemma 7 *Let G be a graph and a* mod n *distribution D on G. For any $u \in V(G)$, there exists a closed walk τ such that*

$$D(\tau)(v) \equiv \begin{cases} D(v) \mod n & \text{for } v \in \mathrm{Link}(u;G) \\ 0 \mod n & \text{for } v \notin \mathrm{Link}(u;G) \cup \{u\} \end{cases}$$

Proof

Let $\{v_1, v_2, \ldots, v_p\}$ be $\mathrm{Link}(u;G)$. For $k = 1, 2, \ldots, p$, define walks τ_k with length 2 by $\tau_k(0) = u, \tau_k(1) = v_k$, and $\tau_k(2) = u$. Then $D(\tau_k)(v_k) \equiv 1 \mod n$ and $D(\tau_k)(v) \equiv 0 \mod n$ for $v \neq u, v_k$. Set $\tau = D(v_1)\tau_1 \# D(v_2)\tau_2 \# \cdots \# D(v_p)\tau_p$. Then we have $\tau(0) = u$, $D(\tau)(v_k) \equiv D(v_k) \mod n$ and $D(\tau)(v) \equiv 0 \mod n$ for $v \neq u, v_1, v_2, \ldots, v_p$.

q.e.d.

Proof of Lemma 4

Note that G is connected. Then there exists a walk ρ such that $\rho(0) = u_0$, $\rho(m_\rho) = u_1$ and $\rho : \{0, 1, \ldots, m_\rho\} \to V(G)$ is surjective, where m_ρ is the length of ρ. For $k = 0, 1, \ldots, m_\rho$, set $T(k) = \cup_{i=0}^{k} \mathrm{Link}(\rho(i); G) - \{\rho(k)\}$. By induction on k, we will prove that, for $k = 0, 1, \ldots, m_\rho$, there exist walks τ_k such that

$$D(\tau_k)(v) \equiv D(v) \mod n \quad \text{for } v \in T(k)$$

and $\tau_k(0) = \rho(0), \tau_k(m_k) = \rho(k)$, where m_k is the length of τ_k. Note that $T(0) = \mathrm{Link}(\rho(0); G)$. Then such τ_0 exists by Lemma 7. Suppose that there exists τ_k as aboves. We define a walk τ'_k such that

$$\tau'_k(i) = \begin{cases} \tau_k(i) & \text{for } 0 \leq i \leq m_k \\ \rho(k+1) & \text{for } i = m_k + 1. \end{cases}$$

Then $D(\tau_k')(v) \equiv D(v) \mod n$ for $v \in T(k)$. By Lemma 7, there exists a closed walk $\bar{\tau}_{k+1}$ such that $\bar{\tau}_{k+1}(0) = \rho(k+1)$ and

$$D(\bar{\tau}_{k+1})(v) \equiv \begin{cases} (D - D(\tau_k'))(v) & \mod n \quad \text{for } v \in \text{Link}(\rho(k+1); G) \\ 0 & \mod n \quad \text{for } v \notin \text{Link}(\rho(k+1); G) \cup \{\rho(k+1)\}. \end{cases}$$

Put $\tau_{k+1} = \tau_k' \# \bar{\tau}_{k+1}$. By Lemma 5, we have
$D(\tau_{k+1})(v) \equiv D(\tau_k')(v) + D(\bar{\tau}_{k+1})(v) \mod n$ for any v. Then
$D(\tau_{k+1})(v) \equiv D(\tau_k')(v) \mod n$ for $v \notin \text{Link}(\rho(k+1); G) \cup \{\rho(k+1)\}$. Therefore
$D(\tau_{k+1})(v) \equiv D(v) \mod n$ for $v \in T(k) - \text{Link}(\rho(k+1); G) \cup \{\rho(k+1)\}$.
Moreover $D(\tau_{k+1})(v) \equiv D(\tau_k')(v) + (D - D(\tau_k'))(v) \equiv D(v) \mod n$
for $v \in \text{Link}(\rho(k+1); G)$.
Then we have $D(\tau_{k+1})(v) \equiv D(v) \mod n$ for $v \in T(k+1)$ Put $\tau = \tau_{m_\rho}$. Note that $T(m_\rho) = V(G) - \{u_1\}$. Then τ such that $\tau(0) = u_0, \tau(m) = u_1$ and $D(\tau)(v) = D(v)$ $(v \neq u_1)$, where m is the length of τ.

q.e.d.

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On the Functional Central Limit Theorem for Walsh series with general gaps

by

K. Ôhashi

ABSTRACT. Let $\{w_n(\omega)\}$ be the Walsh system by Paley order, $\{n_k\}$ be an increasing sequence of the natural numbers and $\{a_k\}$ be the sequence of real numbers with l^2-divergence. From $\{n_k\}$ and $\{a_k\}$, we define the partial sum of Walsh series as $S_k(\omega) = \sum_{j=1}^{k} a_j w_{n_j}(\omega)$ $(k = 1,2,\ldots)$. Using $S_n(\omega)$, we define $X_n : \Omega=[0,1] \to C[0, 1]$ as

$$X_n(t,\omega) = \frac{1}{A_n}\{S_k + a_{k+1}w_{n_{k+1}}\frac{tA_n^2 - A_k^2}{A_{k+1}^2 - A_k^2}\}, \text{ if } \frac{A_k^2}{A_n^2} \le t \le \frac{A_{k+1}^2}{A_n^2} \quad (k = 0,1,\ldots,n-1).$$

This paper shows the set of conditions for the functional central limit theorem of $X_n(t,\omega)$ which includes a new 'diophontine' type condition and with this set of conditions, X_n converges in the sense of distribution to the Wiener process W in $(C, \mathscr{C})$.

§ 1. Introduction.

In this paper we show the Functional Central Limit Theorem(FCLT) for the Walsh series with general gaps. This paper is different from the previous one [6] in the point that the present materials constituing our stochastic processes are not generally martingales. It is easy to see that our theorem is more general than the case in [6]. But it is rather difficult to treat the remainder. Therefore we use the maximal type inequality for the partial sum of the Walsh Fourier series. In [6], we gave the proof of the theorem by using B.M.Brown's theorem. However, from the point of the real analyst's view, it is desired that the proof shall be by the direct proof without Brown's theorem and complicated methods. Therefore, we will give the direct proof in the following.

Let $\{w_n(\omega)\}$ be the Walsh system by Paley order and these functions are defined on the interval [0,1] and only take values of either -1 or +1. Usually these functions are defined as follows. Let $\{r_n(\omega)\}$ be the Rademacher functions , that is,

$r_0(\omega) = r(\omega) = 1$ on [0, 1/2), $r(\omega) = -1$ on [1/2, 1), $r_0(\omega+1) = r_0(\omega)$,

$r_k(\omega) = r(2^k\omega)$ $(k = 1,2,\ldots)$,

and then the Walsh functions are defined as

$$w_0 = 1 \text{ and for } n = \sum_{k=0}^{\infty} \varepsilon_k 2^k,\ w_n = \prod_{k=0}^{\infty} r_k^{\varepsilon_k}.$$

Now, let $\{n_k\}$ be an increasing sequence of the natural numbers and $\{a_k\}$ be the sequence of real numbers such that

$$(1.1) \quad A_n^2 = \sum_{k=1}^{n} a_k^2 \to +\infty \text{ as } n \to \infty.$$

From $\{n_k\}$ and $\{a_k\}$, we define the partial sum of Walsh series as follows;

$$S_k(\omega) = \sum_{j=1}^{k} a_j w_{n_j}(\omega) \quad (k = 1,2,\ldots),$$

and consider them on the probability space (Ω, F, P) ($\Omega = [0,1]$, F = the σ-field of Borel sets in Ω, P = Lebesgue measure on F). Using $S_n(w)$, we define $X_n : \Omega \to C[0, 1]$ as follows:

$$X_n(t,\omega)=\frac{1}{A_n}\{S_k+a_{k+1}w_{n_{k+1}}\frac{tA_n^2-A_k^2}{A_{k+1}^2-A_k^2}\},\ \text{if}\ \frac{A_k^2}{A_n^2}\le t\le\frac{A_{k+1}^2}{A_n^2}.$$

(k= 0,1,...,n -1). Here we call $X_n(t,w)$ the random function on [0,1].

As the preliminary, we see that $X_n(t,w)$ is continuous in [0,1]. Under the conditions below, X_n converges in the sense of distribution to the Wiener process W in (C, $\mathcal{C}$) where C denotes C[0,1] with the sup norm topology and $\mathcal{C}$ the Borel σ-field generated by all open sets in C. To describe our results for this problem , we need the following notation: for two natural numbers n and m, we define the addition n∗m by

$$n*m=\sum_{j=0}^{\infty}|\varepsilon_j-\varepsilon_j'|2^j\quad \text{where } n=\sum_{j=0}^{\infty}\varepsilon_j 2^j\ ,\ m=\sum_{j=0}^{\infty}\varepsilon_j' 2^j\ .$$

Note that $w_n\cdot w_m=w_{n*m}$. Given the sequence $\{n_k\}$ and $\{a_k\}$, we use also the notation,

(1.2) $p(0)=0$, $p(k)=\max\{j;n_j<2^k\}$, $k=1,2,3,...,$

(1.3) $D_{-1}=0$, $D_k=\sum_{j=p(k)+1}^{p(k+1)}a_j w_{n_j}$, $B_k=A_{p(k)}$, $k=0,1,2,....$

Let n be an integer such that

$n=n_k*n_m$, $2^{j-1}\le n_k<n_m<2^j$,and $1\le j$.

Then let E_j (n) be the set of the pairs (n_u, n_v) satisfying $n=n_u*n_v$, $n_u<n_v<2^j$ and let #E (n) denotes the cardinarity of E (n). In § 2 we require such a 'diophantine' type assumption such as

$$\#E_j(n)=o(\phi(j))$$

where o(1) is uniformly in ν and $\phi(j)$ is a nondecreasing function satisfying

(1.4) $\sum_{j=1}^{n}\phi(j+1)(1-p(j)p(j+1)^{-1})B_{j+1}^2(B_{j+1}^2-B_j^2)=O(B_{n+1}^4)$.

In the following, let (C, $\mathcal{C}$, P_W) be the probability space and P_W be the Wiener measure.

§ 2.Results.

For the random function $X_n(t)$ defined in § 1, let $\{P_n\}$ be the sequence of probability measures on (C, $\mathcal{C}$) determined by the distributions of $\{X_n(t,\omega)$, $0\le t\le 1\}$. Under the notations, we show the following theorem;

Theorem.

Let the sequences $\{n_k\}$,$\{a_k\}$ and $\{\phi(k)\}$ satisfy the following conditions

(2.1) $p(j+1)/p(j)\to 1$ as $j\to\infty$,

(2.2) $a_n=O(A_n/\sqrt{n})$ and $A_n\to\infty$ as $n\to\infty$,

and

(2.3) $\#E_j(\nu)=o(\phi(j))$ uniformly in ν,

where $\phi(j)$ satisfies (1.4).

Then we have

$P_n\to P_W$ weakly as $n\to\infty$.

From now on, we abbreviate the conclusion of the theorem as

(2.4) $X_n\Rightarrow W$ as $n\to\infty$. (see[2])

Remark 1.

Our theorem is the best possible in the sense that if we replace $\#E_j(\nu)=o(\phi(j))$ in (2.3) by

(2.5) $\#Ej(\nu) = O(\phi(j))$,
then the theorem becomes false (see[5]).

Corollary.1

Let $\{a_k\},\{n_k\}$ be

(2.6) $n_{k+1}/n_k \geqq 1 + c k^{-\alpha}$ $(0 \leqq \alpha \leqq 1/2,\ c>0)$, $k = 1,2,...$

(2.7) $A_0 = 0$, $An \to \infty$ and $an = o(An/n^{\alpha})$ as $n \to \infty$.

Then we have (2.4) (see [4]).

Remark 2.

The other examples can be found in [5] .

Corollary 2.

If we construct the Xn(t) from the $S_{2^n}(t)$ replaced by the partial sum $S_n(t)$ in § 1, then we hold the same conclusion of the theorem(see[6]).

§ 3. Proposition and Lemmas.

The proposition is a B.M. Brown's theorem for the concrete martingale, but our conditions are a little different from the ones given by him. The lemmas below are for the proof of the following proposition:

Proposition.

Under the conditions of the theorem, for k = 0,1,2,..,n-1, let

(3.1) $$Y_n(t) = \frac{1}{B_n}\{\sum_{j=-1}^{n-1}\Delta_j + \Delta_k \frac{tB_n^2 - B_k^2}{B_{k+1}^2 - B_k^2}\} \quad \text{if} \frac{B_k^2}{B_n^2} \le t \le \frac{B_{k+1}^2}{B_n^2},$$

then we have $Y_n \Rightarrow W$ as $n \to \infty$.

The essence of the proof is to show that Yn has the properties; 1) Yn is tight and 2) the finite dimensional distribution of Yn(t) converges to the finite distribution of W(t).

Here the sequence Yn(t) is said to be tight if for each $\varepsilon > 0$, there is a compact set $K_\varepsilon \in C$, for which

$$P(Y_n^{-1}(K_\varepsilon)) > 1 - \varepsilon \quad \text{for every } n \geqq n_0 .$$

By the Prohorov's theorem (see [1]), this definition is equivalent to the relative compactness of C, thus if we could prove

i) $\forall \eta > 0, \exists a > 0 : P(|Y_n(0)| > a) < \eta$ (n=1,2,...)

ii) $\forall \varepsilon > 0,\ \lim_{\delta \to 0} \limsup_{n \to \infty} P\{\sup_{|t-y|<\delta} |Y_n(t) - Y_n(y)| > \varepsilon\} = 0$

then we would obtain the tightness of the $Y_n(t)$.

By the construction, i) is evident. Thus we prove ii). However for the proof of ii), we need the following lemmas 1-3.

Lemma 1.([5])

Under the conditions of the theorem, we have

(3.2) $$\lim_{n \to \infty} E(|B_n^{-2}\sum_{j=0}^{n-1}\Delta_j^2 - 1|^2) = 0.$$

Lemma2.

Let $\{M_j\}$(j=1,...,n) be the L^1 - martingale sequence. Then the following inequality holds, for $\lambda > 0$,

$$P\{\max_{1 \le j \le n} |M_j| > 2\lambda\} \le \int_{\{|M_n| > \lambda\}} \frac{1}{\lambda} |M_n| \, dP.$$

Proof of Lemma2

We give the simple proof. Let

$$E; = \{\max_{1\le j\le n} | M_j | > 2\lambda\} = \{| M_1 | > 2\lambda\} \cup \bigcup_{j=2}^{n} \{\max_{1\le k<j} | M_k | \le 2\lambda, | M_j | > 2\lambda\}$$

$$= \bigcup_{j=1}^{n} E_j, \text{ say.}$$

Since $\{ M_j \}$ is submartingale,

$$P(E) = \frac{1}{2\lambda}\sum_{j=1}^{n}\int_{E_j} |M_j| \, dP \le \frac{1}{2\lambda}\sum_{j=1}^{n}\int_{E_j} | M_n | \, dP = \int_E \frac{1}{2\lambda} | M_n | \, dP$$

$$= \Big(\int_{E\cap\{|M_n|>\lambda\}} + \int_{E\cap\{|M_n|\le\lambda\}}\Big)\Big(\frac{1}{2\lambda} | M_n | \, dP \le \int_{\{|M_n|>\lambda\}} \frac{1}{2\lambda} | M_n | \, dP + \frac{1}{2} P(E).$$

Hence, $P(E) \le \int_{\{|M_n|>\lambda\}} \frac{1}{\lambda} | M_n | \, dP$. This proves the Lemma 2.

Proof of Proposition Now by (2.1) and (2.2), we have

$$\sum_{j=p(k)+1}^{p(k+1)} a_j^2 = O(B_{k+1}^2 \log \frac{p(k+1)}{p(k)}) = o(B_{k+1}^2) \quad \text{as } k \to \infty.$$

Thus,

(3.3) $\quad B_{n+1}^2 - B_n^2 = o(B_{n+1}^2)$ as $n \to \infty$.

Let us prove the following (3.4) and (3.5) equivalent to ii).

(3.4) for any α_i such that $0 < \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1$,

$$(Y_n(\alpha_0), Y_n(\alpha_1), \ldots, Y_n(\alpha_k)) \xrightarrow{d} (W(\alpha_0), W(\alpha_1), \ldots, W(\alpha_k)),$$

(3.5) for $\varepsilon > 0$, $\quad \lim_{\delta\to 0} \limsup_{n\to\infty} P\{\sup_{|t-y|<\delta} | Y_n(t) - Y_n(y) | > \varepsilon\} = 0$.

In order to prove (3.4), it is sufficient to show

$$(3.6) \quad \begin{aligned} &(Y_n(\alpha_0), Y_n(\alpha_1) - Y_n(\alpha_0), \ldots, Y_n(\alpha_k) - Y_n(\alpha_{k-1})) \\ &\xrightarrow{d} (W(\alpha_0), W(\alpha_1) - W(\alpha_0), \ldots, W(\alpha_k) - W(\alpha_{k-1})). \end{aligned}$$

Now, if we put

$$m_j = \max\{m \ge 0; B_m^2 \le \alpha_j B_n^2\}, \quad (j = 0, 1, 2, \ldots, k),$$

then we have

$$E\{| Y_n(\alpha_j) - Y_n(B_{m_j}^2 / B_n^2) |^2\} \le B_n^{-2} E\Delta_{m_j}^2 = o(1),$$

since (3.3) holds. Hence we have

$$Y_n(\alpha_j) - Y_n(B_{m_j}^2 / B_n^2) \xrightarrow{d} 0 \quad \text{as } n \to \infty.$$

Thus, in the place of proof of (3.6), it is sufficient to see that

$$(3.7) \quad \begin{aligned} &(B_n^{-1} \sum_{j=m_0+1}^{m_1-1} \Delta_j, B_n^{-1} \sum_{j=m_1+1}^{m_2-1} \Delta_j, \ldots, B_n^{-1} \sum_{j=m_{k-1}+1}^{m_k-1} \Delta_j) \\ &\xrightarrow{d} (W(\alpha_1) - W(\alpha_0), W(\alpha_2) - W(\alpha_1), \ldots, W(\alpha_k) - W(\alpha_{k-1})). \end{aligned}$$

However using the technique of H.Cramer and H.Wold, it suffices to give that k-dimensional characteristic function of the left side of (3.7) converges to the characteristic function of the right side. That is, for every $t_1, t_2, \ldots, t_k$, we can simply prove

(3.8) $E\exp\{i\sum_{j=1}^{k} t_j B_n^{-1} \sum_{r=m_{j-1}+1}^{m_j-1} \Delta_r\} \to \exp\{-\frac{1}{2}\sum_{j=1}^{k} t_j^2(\alpha_j - \alpha_{j-1})\}$ as $n \to \infty$.

Now for the proof of (3.8), we set

$$\theta_r = \begin{cases} t_j & \text{if } m_{j-1} < r < m_j \\ 0, & \text{otherwise} \end{cases} \quad (j = 1,2,\dots,k)$$

and

$$\theta = \max_{r \le m} |\theta_r| = \max_{1 \le j \le k} |t_j|,\ D_j = B_n^{-1}\theta_j \Delta_j,\ T_n = \sum_{j=0}^{n-1} D_j$$

and

$$\sigma_n^2 = \sum_{j=1}^{k} t_j^2 (B_{m_j}^2 - B_{m_{j-1}}^2)/B_n^2.$$

Then it is sufficient to prove

(3.9) $$\lim_{n\to\infty} \left| \int_0^1 \exp(iT_n)dP - \exp(-\frac{1}{2}\sigma_n^2) \right| = 0.$$

For, since, by (3.3),

$$0 \le \alpha_j - B_{m_j}^2 / B_n^2 \le (B_{m_j+1}^2 - B_{m_j}^2)/B_n^2 = o(1) \text{ as } n \to \infty.$$

Following D.L.McLeish' s method [3], let us put

$$Z_j = D_j I\{\sum_{r=0}^{j-1} D_r^2 \le 2\theta^2\},\ j = 0,1,2,\dots,n-1$$

$$E_n = \{Z_j \ne D_j \text{ for some } j < n\},$$

then we have, by Lemma 1,

(3.10) $$P(E_n) \le P\{\sum_{j=0}^{n-1} D_j^2 > 2\theta^2\} \le P\{B_n^{-2}\theta^2 \sum_{j=0}^{n-1} \Delta_j^2 > 2\theta^2\}$$

$$= P(B_n^{-2}\sum_{j=0}^{n-1}\Delta_j^2 - 1 > 1) \le \frac{1}{1^2} E(|B_n^{-2}\sum_{j=0}^{n-1}\Delta_j^2 - 1|^2)$$

$$\to 0 \text{ as } n \to \infty.$$

Next, if we set

$$J_n = \begin{cases} \min\{j \le n-1; \sum_{i=0}^{j} D_i^2 > 2\theta^2\} & \text{if } \sum_{j=0}^{n-1} D_i^2 > 2\theta^2 \\ n-1, & \text{otherwise,} \end{cases}$$

then we have

(3.11) $$\int_0^1 \left|\prod_{j=0}^{n-1}(1+iZ_j)\right|^2 dP = \int_0^1 \prod_{j=0}^{n-1}(1+Z_j^2)dP$$

$$\le \int_0^1 \exp\{\sum_{j=0}^{J_n-1} D_j^2\}(1+D_{J_n}^2) \le e^{2\theta^2}(1+\int D_{J_n}^2 dP)$$

$$\le e^{2\theta^2}(1+\theta^2 B_n^{-2}\int_0^1 \sum_{j=0}^{n-1}\Delta_j^2 dP) = e^{2\theta^2}(1+\theta^2).$$

To see (3.9), since by (3.10),

$$\left|\int_{E_n} e^{iT_n} dP\right| \le P(E_n) \to 0 \text{ as } n \to \infty,$$

it remains to estimate the expectation of $\exp(iT_n)$ on E_n^c.

Noting $|Z_j| \leq \sqrt{2}\theta$ on E_n^c, we use the formula

$$e^{it} = (1+it)\exp\{-\frac{t^2}{2} + O(|t|^3)\}$$

if t is in the bounded range. The following estimate holds:

$$\int_{E_n^c} e^{iT_n} dP = \int_{E_n^c} \prod_{j=0}^{n-1}(1+iZ_j)\exp\{-\frac{1}{2}\sum_{j=0}^{n-1} Z_j^2 + O(\sum_{j=0}^{n-1}|Z_j|^3)\}dP$$

$$= \int_{E_n^c} \prod_{j=0}^{n-1}(1+iZ_j)\exp(-\frac{1}{2}\sum_{j=0}^{n-1} Z_j^2)dP$$

$$+ \int_{E_n^c} \prod_{j=0}^{n-1}(1+iZ_j)\exp(-\frac{1}{2}\sum_{j=0}^{n-1} Z_j^2)\{\exp O(\sum_{j=0}^{n-1}|Z_j|^3) - 1\}dP$$

$$= I_1(n) + I_2(n), \text{ say.}$$

For $I_2(n)$, if we note that the absolute value of the integrand is $\leqq 2$, then it is enough to show that

$$\sum_{j=0}^{n-1}|Z_j|^3 \to 0 \ (n \to \infty) \text{ in probability}.$$

For, since

$$\sum_{j=0}^{n-1}|Z_j|^3 \leq \max_{0\leq j\leq n-1}|Z_j| \sum_{j=0}^{n-1} Z_j^2 \leq 2\theta \max_{0\leq j\leq n-1}|Z_j|$$

$$\leq 2\theta \frac{\theta}{B_n} \max_{0\leq j\leq n-1}|\Delta_j|,$$

we have

$$(3,12) \quad P(\varepsilon < \max_{0\leq j\leq n-1}\frac{1}{B_n}|\Delta_j|) = P\{\varepsilon^2 < \sum_{j=0}^{n-1}\frac{1}{B_n^2}\Delta_j^2 I(B_n^{-1}|\Delta_j| > \varepsilon)\}$$

$$\leq \varepsilon^{-2}\sum_{j=0}^{n-1} B_n^{-2}\int_0^1 \Delta_j^2 I(B_n^{-1}|\Delta_j| > \varepsilon)dP$$

$$\leq \varepsilon^{-2}\sum_{j=0}^{n-1} B_n^{-2}(\int_0^1 \Delta_j^4 dP)^{1/2}(P(B_n^{-1}|\Delta_j| > \varepsilon))^{1/2}$$

$$\leq \varepsilon^{-2}\sum_{j=0}^{n-1} B_n^{-2}(\int_0^1 \Delta_j^4 dP)^{1/2}(\varepsilon^{-2}B_n^{-2}E\Delta_j^2)^{1/2}$$

$$\leq \varepsilon^{-3}B_n^{-3}(\sum_{j=0}^{n-1}\int_0^1 \Delta_j^4 dP)^{1/2}(\sum_{j=0}^{n-1}E\Delta_j^2)^{1/2} = \varepsilon^{-3}B_n^{-2}(\sum_{j=0}^{n-1}\int_0^1 \Delta_j^4 dP)^{1/2}$$

$$= \varepsilon^{-3}B_n^{-2}[\sum_{j=0}^{n-1}\{o(B_j^2(B_{j+1}^2 - B_j^2)) + o(\phi(j+1)B_{n-1}^2(1-p(j)p(j+1)^{-1})E\Delta_j^2\}]^{1/2}$$

$$= o(1) \quad \text{as } n \to \infty, \text{ by (1.4). (see (3.3) in [5])}$$

Thus, $I_2(n) \to 0$ as $n \to \infty$. For $I_1(n)$, by the mean-value theorem and Minkowski inequality,

$$I_1(n) = \int_{E_n^c} \prod_{j=0}^{n-1} (1 + iZ_j) e^{-\frac{1}{2}\sigma_n^2} dP$$

$$+ \int_{E_n^c} \prod_{j=0}^{n-1} (1 + iZ_j)\{\exp(-\frac{1}{2}\sum_{j=0}^{n-1} Z_j^2) - \exp(-\frac{1}{2}\sigma_n^2)\} dP$$

$= I_{11}(n) + I_{12}(n)$, say .

Now, for $I_{12}(n)$,

$$\le \frac{1}{2} e^{\theta^2} (1+\theta^2)^{1/2} \left(\int_0^1 | \sum_{j=0}^{n-1} Z_j^2 - \sigma_n^2 |^2 dP\right)^{1/2}$$

$$\le C\{\int_0^1 | \sum_{j=0}^{n-1} \theta_j^2 B_n^{-2} (\Delta_j^2 - E\Delta_j^2) |^2)\}^{1/2}$$

$$\le C\theta \sum_{j=0}^{k} \{\int_0^1 | B_n^{-2} \sum_{r=m_{j-1}+1}^{m_j-1} (\Delta_r^2 - E\Delta_r^2) |^2\}^{1/2} \to 0 \text{ as } n \to \infty,$$

by Lemma 1. Next, for $I_{11}(n)$,

$$I_{11}(n) = \int_0^1 \prod_{j=0}^{n-1} (1 + iZ_j) \exp(-\frac{1}{2}\sigma_n^2) dP - \int_{E_n} \prod_{j=0}^{n-1} (1 + iZ_j) \exp(-\frac{1}{2}\sigma_n^2) dP$$

$$= \exp(-\frac{1}{2}\sigma_n^2) - \int_{E_n} \prod_{j=0}^{n-1} (1 + iZ_j) \exp(-\frac{1}{2}\sigma_n^2) dP,$$

since Z_j is martingale difference. From (3.10) and (3.11),

$$| \int_{E_n} \prod_{j=0}^{n-1} (1 + iZ_j) \exp(-\frac{1}{2}\sigma_n^2) dP |$$

$$\le e^{\theta^2} (1+\theta^2)^{1/2} \exp(-\frac{1}{2}\sigma_n^2) P(E_n)^{1/2} \to 0 \text{ as } n \to \infty.$$

Combining these estimates, we have (3.9) and thus (3.4) has been proven.

Next we show (3.5). We take any given $\varepsilon > 0$ and fix it.

$$P\{\sup_{|t-y|\le\delta} | Y_n(t) - Y_n(y) | > \varepsilon\} \le \sum_{k\delta\le 1} P\{\sup_{k\delta\le t\le(k+1)\delta} | Y_n(t) - Y_n(k\delta) | > \frac{\varepsilon}{3}\}$$

and set

$$q_0 = 0, \quad q_k = \max\{j \ge 1 : B_j^2 \le k\delta B_n^2\} \ (k = 1, 2, \ldots, [\frac{1}{\delta}] + 1).$$

The definition of $Y_n(t)$ implies

$$\sup_{k\delta\le t\le(k+1)\delta} | Y_n(t) - Y_n(k\delta) | \le \max_{q_k \le r \le q_{k+1}} | \sum_{j=q_k}^{r-1} \Delta_j / B_n | + 2 \max_{1\le j\le n} | \Delta_j / B_n |$$

for sufficiently large n. Hence we have

$$P\{\sup_{|t-y|\le\delta} | Y_n(t) - Y_n(y) | > \varepsilon\} \le \sum_{k\delta<1} P\{\max_{q_k\le r\le q_{k+1}} | \sum_{j=q_k}^{r-1} \Delta_j / B_n | > \varepsilon/6\}$$

$$+ \sum_{k\delta<1} P\{\max_{1\le j\le n} \frac{|\Delta_j|}{B_n} > \frac{\varepsilon}{12}\}$$

$= J_1(n) + J_2(n)$, say.

By (3.12), $J_2(n) \to 0$ as $n \to \infty$. For $J_1(n)$, if we put $M_s = \sum_{j=0}^{s} \Delta_j$, then we have

$$\sum_{j=q_k}^{r-1} \Delta_j = M_{r-1} - M_{q_k-1},$$

and $\{M_s\}$ is martingale, because of the σ-field generated by the suitable Rademacher functions. By Lemma 2, we obtain

$$J_1(n) \le \sum_{k\delta<1} E\{\frac{12}{\varepsilon} B_n^{-1} | M_{q_{k+1}} - M_{q_k-1} |; | M_{q_{k+1}-1} - M_{q_k-1} | > \frac{\varepsilon}{12} B_n\}$$

$$\le \delta + \sum_{k\delta<1} E\{\frac{12}{\varepsilon} | W((k+1)\delta) - W(k\delta) |; | W((k+1)\delta) - W(k\delta) | \ge \frac{\varepsilon}{12}\},$$

for sufficiently large n, using the main theorem in [5].
Thus, we have

$$J_1(n) \le \delta + \sum_{k\delta<1} 2 \cdot \frac{12}{\varepsilon} \int_{\varepsilon/12}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\delta}} \exp(-x^2/2\delta) dx$$

$$\le \delta + \frac{24}{\varepsilon} \cdot \frac{1}{\delta} \cdot \sqrt{\delta} \exp(-\varepsilon^2/288\delta) \to 0 \text{ as } \delta \to 0.$$

Therefore, (3.5) is completely proven. Combining (3.4) and (3.5), we have $Y_n \Rightarrow W$ as $n \to \infty$.

§ 4. Proof of Theorem.

Now for any given N, we can find an integer M = M(N) such that $p(M) < N \leqq p(M+1)$ and define the random function $W_N(t)$ by

$$(4.1) \quad W_N(t) = \frac{1}{A_N}\{S_{p(k)} + \Delta_k \frac{tA_N^2 - A_{p(k)}^2}{A_{p(k+1)}^2 - A_{p(k)}^2}\}, \text{if } \frac{A_{p(k)}^2}{A_N^2} \le t \le \frac{A_{p(k+1)}^2}{A_N^2}$$

(k=0,1,2,...,M). Then to prove the theorem it is sufficient to show that for any $\varepsilon > 0$,

$$(4.2) \quad P\{\max_{0\le t\le 1} | X_N(t) - Y_{M+1}(t) | \ge \varepsilon\} \to 0 \text{ as } N \to \infty.$$

We have

$$\max_{0\le t\le 1} | X_N(t) - Y_{M+1}(t) | \le \max_{0\le t\le 1} | X_N(t) - W_N(t) |$$

$$+ \max_{0\le t\le 1} | W_N(t) - \frac{A_{p(M+1)}}{A_N} Y_{M+1}(t) | + (\frac{A_{p(M+1)}}{A_N} - 1) \max_{0\le t\le 1} | Y_{M+1}(t) |$$

$$= K_{1,N} + K_{2,N} + K_{3,N}, \text{ say.}$$

By the Proposition, $P_n = PY_n^{-1}$ $(n = 1,2,...)$ are relatively compact, and by the well-known Prohorov's theorem, $\{P_n\}$ is tight. Therefore by the proposition that is equivalent to tightness (c.f.[1]), for every $\eta > 0$ and $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that

$$(4.3) \quad P\{\max_{|t-s|<\delta} | Y_n(t) - Y_n(s) | \ge \frac{\varepsilon}{6}; \ t,s \in [0,1]\} < \eta \text{ if } n \ge n_0.$$

On the other hand, (3.2) implies that $| A_N^2 / A_{p(M+1)}^2 - 1 | < \delta$ if $N \ge N_0$.
Thus we have

$$\max_{0\le t\le 1} | W_N(t) - \frac{A_{p(M+1)}}{A_N} Y_{M+1}(t) | = \frac{A_{p(M+1)}}{A_N} \max_{0\le t\le 1} | Y_{M+1}(\frac{A_N^2}{A_{p(M+1)}} t) - Y_{M+1}(t) |$$

$$\le 2 \max_{|s-t|<\delta} \{ | Y_{M+1}(s) - Y_{M+1}(t) |;\ s,t \in [0,1] \} \quad \text{if } N \ge N_0$$

and by (4.3),

$$P(| K_{2,N} | \ge \frac{\varepsilon}{3}) \le P\{ (\max_{|t-s|<\delta} | Y_{M+1}(t) - Y_{M+1}(s) |;\ t,s \in [0,1]) \ge \frac{\varepsilon}{6} \} < \eta$$

if $N \ge \max(p(n_0), N_0)$.

Since $\max_{0\le t\le 1} | Y_{M+1}(t) | \le \max_{0\le k\le M+1} (| S_{p(k)} | + | \Delta_k |) / B_{M+1}$, we have by the submartingale inequality and (3.12),

$$P(| K_{3,N} | \ge \frac{\varepsilon}{3}) \le P\{ \max_{0\le k\le M+1} | S_{p(k)} | \ge \frac{\varepsilon}{6} \cdot B_{M+1} (\frac{A_{p(M+1)}}{A_N} - 1)^{-1} \}$$

$$+ P\{ \max_{0\le k\le M+1} \frac{| \Delta_k |}{B_{M+1}} \ge \frac{\varepsilon}{6} (\frac{A_{p(M+1)}}{A_N} - 1)^{-1} \}$$

$$\le (\frac{\varepsilon}{6} \cdot B_{M+1})^{-2} (\frac{A_{p(M+1)}}{A_N} - 1)^2 E S_{p(M+1)}^2 + o(1) = o(1) \quad \text{as } N \to \infty.$$

Finally we estimate $P(| K_{1,N} | > \varepsilon / 3) \to 0$ as $N \to \infty$. If we put

$$M_k(x) = \max_{p(k)<r<p(k+1)} | \sum_{j=p(k)+1}^{r} a_j w_{n_j}(x) / A_N |,$$

then we have $K_{1,N} \le \max_{1\le k\le M} | \Delta_k | / A_N + \max_{0\le k\le M} M_k$ and thus

$$P(| K_{1,N} | > \varepsilon / 3) \le P(\max_{1\le k\le M} | \Delta_k | / A_N > \varepsilon / 6) + P(\max_{0\le k\le M} M_k > \varepsilon / 6)$$

$$= L_N(1) + L_N(2).$$

$L_N(1)$ is same as the second part estimate in $K_{3,N}$. Thus $L_N(1) \to 0$ as $N \to \infty$. For $L_N(2)$,

$$L_N(2) = P\{ (\frac{\varepsilon}{6})^2 < \sum_{k=0}^{M} M_k^2 I(M_k > \frac{\varepsilon}{6}) \} \le (\frac{\varepsilon}{6})^{-2} \sum_{k=0}^{M} E\{ M_k^2 I(M_k > \frac{\varepsilon}{6}) \}$$

$$\le (\frac{\varepsilon}{6})^{-2} \sum_{k=0}^{M} (E M_k^2)^{1/2} (P(M_k > \frac{\varepsilon}{6}))^{1/2},$$

for the last part, we use Carleson-Hunt almost everywhere convergence theorem,

$$L_N(2) \le (\frac{\varepsilon}{6})^{-2} \cdot C \sum_{k=0}^{M} \{ A_N^{-4} E(\sum_{j=p(k)}^{p(k+1)-1} a_j w_{n_j})^4 \}^{1/2} \{ (\frac{\varepsilon}{6})^{-2} E\Delta_k^2 / A_N^2 \}^{1/2}$$

$$\le (\frac{\varepsilon}{6})^{-3} \cdot C A_N^{-3} (\sum_{k=0}^{M} E\Delta_k^4)^{1/2} (\sum_{k=0}^{M} E\Delta_k^2)^{1/2}$$

$$= (\frac{\varepsilon}{6})^{-3} \cdot C A_N^{-2} (\sum_{k=0}^{M} E\Delta_k^4)^{1/2} (B_{M+1} / A_N) \le C' A_N^{-2} (\sum_{k=0}^{M} E\Delta_k^4)^{1/2}$$

$$= o(1) \quad \text{as } n \to \infty,$$

by (1.4) (see (3.3) in [5]). Thus we have proven our theorem.

Proof of Corollary .

The conditions (2.6) and (2.7) imply Lemma 1. But it is the same lemma in S. Takahashi [7]. Since it is easy to show the main part of the proof, we omit it.

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A Generalization Of Eigenvalue Problems For Systems Of Second Order Linear Differential Equations

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0. Introduction

In this paper we consider a generalization of eigenvalue problems which contains two parameters λ, μ. This problem was considered in Zachmann[8]. The properties of relation between these parameters have been considered when (λ, μ) is an eigenvalue in [8]. In this paper we consider the analytic properties of relation between these parameters λ, μ more precisely when (λ, μ) is an eigenvalue. The definition of eigenvalue will be given later in this paper.

1.Differential Operator and It's Boundary Condition

In this paper we consider the following second order differential operator:

$$L = \begin{pmatrix} p(x) & D^2 + q(x) \\ D^2 + q(x) & r(x) \end{pmatrix}, \text{where } D^2 = \frac{d^2}{dx^2}, \tag{1.1}$$

and let $U, V, \cdots$ be two-dimensional vectors $U = (u, v)^T$,etc. The following equation will be considered:

$$(L-\Lambda)U=0, \tag{1.2}$$

where $\Lambda=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$. This equation is equivalent to following system:

$$\begin{cases}\dfrac{d^2}{dx^2}v+q(x)v+p(x)u-\lambda u=0,\\[2ex] \dfrac{d^2}{dx^2}u+q(x)u+r(x)v-\mu v=0,\end{cases} \tag{1.3}$$

where λ and μ are real or complex parameters. $p(x), q(x)$ and $r(x)$ are real valued continuous functions in $[a,b]$.

We define the boundary condition for differential operator L at $x=a, x=b$ by following equalities:

$$\begin{cases}M(a,U)=a_{j1}u(a)+a_{j2}u'(a)+a_{j3}v(a)+a_{j4}v'(a)=0\\ N(b,U)=b_{j1}u(b)+b_{j2}u'(b)+b_{j3}v(b)+b_{j4}v'(b)=0.\end{cases} \quad (j=1,2) \tag{1.4}$$

(1) a_{ij} and b_{ij} are real-valued constants and we assume

$$a_{12}/a_{22} \neq a_{14}/a_{24},\ b_{12}/b_{22} \neq b_{14}/b_{24}. \tag{1.5}$$

(2) a_{ij} and b_{ij} are independent of λ and μ.

(3) We impose following condition:

$$\begin{cases}a_{12}a_{23}+a_{14}a_{21}-a_{11}a_{24}-a_{13}a_{22}=0,\\[1ex] b_{12}b_{23}+b_{14}b_{21}-b_{11}b_{24}-b_{13}b_{22}=0,\end{cases} \tag{1.6}$$

We call vector $U=(u,v)$, a solution of (1.2), an eigenfunction when it satisfies (1.4) for some Λ. And we call such Λ an eigenvalue and we express it by $(\lambda(\mu),\mu)$ or $(\lambda,\mu(\lambda))$. We do not distinguish $(\lambda(\mu),\mu)$ from $(\lambda,\mu(\lambda))$.

Chakravarty[2] studied expansion problems when $\lambda=\mu$. Hilbert[5], Kodaira[6], Neumark[7] and Coddington-Levinson[4] considered eigenvalue problems for matrix differential operator when $\lambda=\mu$. In this paper we consider eigenvalue problems for the matrix differential operator L when λ and μ are given independently. So that our result is different from that of above authors. Zachmann[8] studied an eigenvalue problem when λ and μ are given independently. Zachmann[8] showed that $\lambda=\lambda(\mu)$ is continuous when (λ,μ) is an eigenvalue. We consider the analytic properties of $\lambda=\lambda(\mu)$ more precisely than Zachmann [8], Baghat[1] and Chakravarty[2] when (λ,μ) is an eigenvalue. Our method is different from that of these authors.

2.Boundary condition vectors and bilinear form

Let $\phi_l=\phi_l(x)=\{x_l,y_l\}$, $\phi_j=\phi_j(x)=\{x_j,y_j\}$ be two dimensional vectors, where x_l,y_l,x_j,y_j are functions of x. We define P_{ij} or $[\phi_i,\phi_j]$ by following:

$$P_{ij}=[\phi_i,\phi_j]=\begin{vmatrix} x_i & x_j \\ y_i' & y_j' \end{vmatrix}+\begin{vmatrix} x_i' & x_j' \\ y_i & y_j \end{vmatrix}.$$

This is called a bilinear form of two vectors ϕ_i and ϕ_j. Then we have following:

(1) $[\phi_i,\phi_j]=-[\phi_j,\phi_i]$,

(2) $[\phi_i,\phi_i]=0$,

(3) $[\phi_i,\alpha\phi_j+\beta\phi_k]=\alpha[\phi_i,\phi_j]+\beta[\phi_i,\phi_k]$, where α,β are constants,

(4) If ϕ_i,ϕ_j are solutions of (1.2) or (1.3) for same (λ,μ) then $[\phi_i,\phi_j]$ is independent of x but depends on (λ,μ).

We define $Q(\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5,\Theta_6)$ by following:

$$\begin{aligned}
&Q(\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5,\Theta_6)\\
&=P_{12}(P_{34}P_{56}-P_{35}P_{46}+P_{36}P_{45})-P_{13}(P_{24}P_{56}-P_{25}P_{46}+P_{26}P_{45})\\
&+P_{14}(P_{23}P_{56}-P_{25}P_{36}+P_{26}P_{35})-P_{15}(P_{23}P_{46}-P_{24}P_{36}+P_{26}P_{34})\\
&+P_{16}(P_{23}P_{45}-P_{24}P_{35}+P_{25}P_{34}),
\end{aligned}$$

where $\Theta_i=(\phi_i,\phi_i')^T$ ($i=1,2,\ldots,6$) are 4-dimensional vectors.

$Q(\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5,\Theta_6)$ has following properties:

(i) $Q(\Theta_1,\ldots,\Theta_i,\ldots,\Theta_j,\ldots,\Theta_6)=-Q(\Theta_1,\ldots,\Theta_j,\ldots,\Theta_i,\ldots,\Theta_6)$

(ii) $Q(\Theta_1,\ldots,\Theta_i,\ldots,\Theta_i,\ldots,\Theta_6)=0$

(iii) $Q(\Theta_1,\ldots,\alpha\Theta_i+\beta\Theta_i^*,\ldots,\Theta_i,\ldots,\Theta_6)$

$=\alpha Q(\Theta_1,\ldots,\Theta_i,\ldots,\Theta_j,\ldots,\Theta_6)+\beta Q(\Theta_1,\ldots,\Theta_i^*,\ldots,\Theta_j,\ldots,\Theta_6)$,

where α,β are constants.

We note that for every six vectors Θ_j ($j=1,2,\ldots,6$) we have following identity:

$$Q(\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5,\Theta_6)=0. \tag{2.2}$$

We show this identity. If Θ_j (j=1,2,...,4) are linearly independent from each other, they are fundamental system in 4-dimensional vector space which we consider. So that Θ_5 and Θ_6 will be expressed by linear combination of Θ_j (j=1,2,...,4). Thus by use of above properties (i) , (ii) we have equality (2.2). Even if Θ_j (j=1,2,3,4) are not linearly independent by use of above (i) and (ii) we have (2.2). Thus we can show (2.2).

Take solutions $\phi_l = (u_l(a;x,\Lambda), v_l(a;x,\Lambda))^T$ (l=1,2), $\phi_l = (u_j(b;x,\Lambda), v_j(b;x,\Lambda))^T$ (j=3,4) of (1.2) which satisfy the initial conditions:

$$\begin{aligned}
&u_1(a;a,\Lambda) = -a_{14}, v_1(a;a,\Lambda) = -a_{12}, u_1'(a;a,\Lambda) = a_{13}, v_1'(a;a,\Lambda) = a_{11},\\
&u_2(a;a,\Lambda) = -a_{24}, v_2(a;a,\Lambda) = -a_{22}, u_2'(a;a,\Lambda) = a_{23}, v_2'(a;a,\Lambda) = a_{21},\\
&u_3(b;b,\Lambda) = -b_{14}, v_3(b;b,\Lambda) = -b_{12}, u_3'(b;b,\Lambda) = b_{13}, v_3'(b;b,\Lambda) = b_{11},\\
&u_4(b;b,\Lambda) = -b_{24}, v_4(b;b,\Lambda) = -b_{22}, u_4'(b;b,\Lambda) = b_{23}, v_3'(b;b,\Lambda) = b_{21}.
\end{aligned}$$

These solutions exist in the interval $[a,b]$ by the existence theorem because (1.2) is a system of liner ordinary differential equations. And these solutions are called boundary condition vectors at $x = a$ and $x = b$ respectively.

We represent the boundary condition (1.4) by use of boundary condition vectors $\phi_l = (u_l(a;x,\Lambda), v_l(a;x,\Lambda))^T$ (l=1,2), $\phi_j = (u_j(b;x,\Lambda), v_j(b;x,\Lambda))^T$ (j=3,4). Let $U(\xi,x)=(u(\xi,x), v(\xi,x))^T$ be the vector in (1.4), where $u(\xi,x), u'(\xi,x), v(\xi,x), v'(\xi,x)$ have given value at $x=\xi$ $(a \le \xi \le b)$. The expression of the boundary condition by use of boundary condition vectors is given as following:

$$\begin{cases} M(a,U) = [U(a,x), \phi_l(a;x,\Lambda)](a) = [U,\phi_l](a) = 0 (l = 1,2) \\ N(b,U) = [U(b,x), \phi_j(b;x,\Lambda)](b) = [U,\phi_j](b) = 0 (j = 3,4) \end{cases} \tag{2.3}$$

$$[\phi_1, \phi_2](a) = 0, [\phi_3, \phi_4](b) = 0 . \tag{2.4}$$

Notice that (2.4) and (2.3) are equivalent to (1.6), (1.4) respectively. Following are important:

(i) $\phi_l(a;a,\Lambda)$ (l=1,2) and $\phi_j(b;b,\Lambda)$ (j=3,4) are independent of Λ.

This is verified by definition of $\phi_1, \phi_2, \phi_3, \phi_4$.

(ii) $[\phi_1, \phi_2](x)$ and $[\phi_3, \phi_4](x)$ are independent of x and Λ.

Considering $(d/dx)[\phi_1, \phi_2](x) \equiv 0$ and $(d/dx)[\phi_3, \phi_4](x) \equiv 0$,then we can prove that $[\phi_1, \phi_2](x)$ and $[\phi_3, \phi_4](x)$ are independent of x. Thus $[\phi_1, \phi_2](x)$= $[\phi_1, \phi_2](a)$=0 =$[\phi_3, \phi_4](x)$=$[\phi_3, \phi_4](b)$. This means $[\phi_1, \phi_2](a)$,$[\phi_3, \phi_4](b)$ are independent of Λ. Thus $[\phi_1, \phi_2](x)$,$[\phi_3, \phi_4](x)$ are independent of x , Λ.

(iii) $\phi_l(a;a,\Lambda)$ (l=1,2) are independent of each other and $\phi_j(b;b,\Lambda)$ (j=3,4) are so .

This is verified by the initial conditions of $\phi_l(a;a,\Lambda)$ (l=1,2), $\phi_j(b;b,\Lambda)$ (j=3,4) and (1.5).

[Theorem 1] Let $U(x)$ =$(u_1(x), v_1(x))^T$ *and* $V(x)$ =$(u_2(x), v_2(x))^T$ *satisfy the boundary conditions* $M(a,U) = 0$ *and* $N(a,V) = 0$. *Then* $[U,V](a) = 0$. *Let above* $U(x)$ *and* $V(x)$ *satisfy the boundary conditions* $M(b,U) = 0$ *and* $N(b,V) = 0$. *Then* $[U,V](b) = 0$.

(Proof) By the condition we have $[U,\phi_1](a)$=0=$[U,\phi_2](a)$ and $[V,\phi_1](a)$=0=$[V,\phi_2](a)$, where $[\phi_1,\phi_2](a)=0$. Put $U(x)=\phi_5(x)$, $V(x)=\phi_6(x)$ then we have $P_{12}=0, P_{15}=0, P_{16}=0, P_{25}=0$ and $P_{26}=0$. We have following identity:

$$Q(\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5,\Theta_6)=P_{14}P_{23}P_{56}-P_{13}P_{24}P_{56}=P_{56}(P_{14}P_{23}-P_{13}P_{24})=0.$$

This identity holds for every Λ. $\phi_l(a;a,\Lambda)$ (l=1,2) are linearly independent from each other. We can take $\phi_j(b;x,\Lambda)$ (j=3,4) such that $\phi_1(a;a,\Lambda)$, $\phi_2(a;a,\Lambda)$, $\phi_3(b;a,\Lambda)$ and $\phi_4(b;a,\Lambda)$ are linearly independent from each other for given Λ .Then Wronskian $W(\phi_1,\phi_2,\phi_3,\phi_4)=P_{14}P_{23}-P_{13}P_{24}\neq 0$ ($W(\phi_1,\phi_2,\phi_3,\phi_4)$ will be given later). Noticing that P_{56} dose not depend on choice of $\phi_1,\phi_2,\phi_3,\phi_4$,P_{56} must vanish for all Λ. Thus P_{56}=$[U(x),V(x)](a)=0$. Similarly $[U(x),V(x)](b)=0$. Thus theorem is proved. □

3.Inner product of vectors and Green's theorem

We use the following symbols:

(i) $(y,z)=y_1z_1+y_2z_2$ where $y=(y_1,y_2)^T, z=(z_1,z_2)^T$,

(ii) $\langle y,z\rangle=\int_a^b(y_1z_1+y_2z_2)dx$,

especially $\|y\|=\langle y,y\rangle$.

Suppose that two vectors $F(x)=(F_1(x),F_2(x))^T, G(x)=(G_1(x),G_2(x))^T$ have continuous derivative up to second order. Then we have the Green's formula:

$$\langle F,LG\rangle-\langle G,LF\rangle=[F,G](b)-[F,G](a). \tag{3.1}$$

This is shown by calculation directly. We consider an eigenvalue $\Lambda_1 = (\lambda_1(\mu), \mu)$ (or $(\lambda, \mu_1(\lambda))$) and $\Lambda_2 = (\lambda_2(\mu), \mu)$ (or $(\lambda, \mu_2(\lambda))$). We say $\Lambda_1 = \Lambda_2$ if and only if $\lambda_1(\mu) = \lambda_2(\mu)$ (or $\mu_1(\lambda) = \mu_2(\lambda)$). By use of (3.1) we have following result:

(a) Let $U_1 = (u_1, v_1), U = (u_2, v_2)$ be the solution of (1.3), (1.4) for the different eigenvalue Λ_1, Λ_2 respectively. Then $\langle U_1, U_2 \rangle = 0$.

This is verified as following. For μ we take $\lambda_1(\mu) \neq \lambda_2(\mu)$, why $\Lambda_1 \neq \Lambda_2$. By use of Green's formula we have following relation :

$$\langle U_1, LU_2 \rangle - \langle U_2, LU_1 \rangle = [U_1, U_2](b) - [U_1, U_2](a).$$

$U_1(x), U_2(x)$ satisfy the given boundary condition at $x = a, x = b$. So that $[U_1, U_2](b) = 0 = [U_1, U_2](a)$ by [Theorem 1], and we have $\langle U_1, LU_2 \rangle - \langle U_2, LU_1 \rangle = 0$.

By the property of inner-product (ii) and some calculation we have following relation:

$$\langle U_1, LU_2 \rangle - \langle U_2, LU_1 \rangle = (\lambda_2(\mu) - \lambda_1(\mu)) \int_a^b u_1(x) u_2(x) dx = 0 .$$

By the condition $\lambda_2(\mu) - \lambda_1(\mu) \neq 0$, we have $\int_a^b u_1(x) u_2(x) dx = 0$. Similarly, if we take $\mu_1(\lambda)$ and $\mu_2(\lambda)$ we have $\int_a^b v_1(x) \cdot v_2(x) dx = 0$. Thus we have, $\langle U_1, U_2 \rangle = 0$, the orthogonal relation between $U_1(x)$ and $U_2(x)$.

We express $\phi_l(a; x, \Lambda)$ (l =1,2) and $\phi_j(b; x, \Lambda)$ (j =3,4) as following:

$$\phi_l(a;x,\Lambda)=(u_l(a;x,\lambda,\mu),v_l(a;x,\lambda,\mu))=\phi_l(a;x,\lambda.,\mu)\ (l=1,2)$$

$$\phi_j(b;x,\Lambda)=(u_j(b;x,\lambda,\mu),v_j(b;x,\lambda,\mu))=\phi_j(b;x,\lambda,\mu)\ (j=3,4).$$

Let $\lambda_1 \neq \lambda_2$, then we have following relation by calculation:

$$\langle\phi_l(a;x,\lambda_1,\mu),L\phi_j(b;x,\lambda_2,\mu)\rangle-\langle\phi_j(b;x,\lambda_2,\mu),L\phi_l(a;x,\lambda_1,\mu)\rangle$$

$$=[\phi_l(a;x,\lambda_1,\mu),\phi_j(b;x,\lambda_2,\mu)](b)\ -[\phi_l(a;x,\lambda_1,\mu),\phi_j(b;x,\lambda_2,\mu)](a)$$

$$=[\phi_l(a;x,\lambda_1,\mu),\phi_j(b;x,\lambda_1,\mu)](b)-[\phi_l(a;x,\lambda_2,\mu),\phi_j(b;x,\lambda_2,\mu)](a)$$

$$-[\phi_l(a;x,\lambda_1,\mu),\{\phi_j(b;x,\lambda_1,\mu)-\phi_j(b;x,\lambda_2,\mu)\}](b)$$

$$-[\{\phi_l(a;x,\lambda_1,\mu)-\phi_l(a;x,\lambda_2,\mu)\},\phi_j(b;x,\lambda_2,\mu)](a).$$

$[\phi_l(a;x,\lambda_1,\mu),\phi_j(b;x,\lambda_1,\mu)](x)$ and $[\phi_l(a;x,\lambda_2,\mu),\phi_j(b;x,\lambda_2,\mu)](x)$ do depend only on (λ_1,μ) or (λ_2,μ) respectively and we denote them $P_{lj}(\lambda_1,\mu)$ or $P_{lj}(\lambda_2,\mu)$. Notice, for all (λ_1,μ) and (λ_2,μ), $\{\phi_j(b;b,\lambda_1,\mu)-\phi_j(b;b,\lambda_2,\mu)\}=0$, $\{\phi_l(a;a,\lambda_1,\mu)-\phi_l(a;a,\lambda_2,\mu)\}$ $=0$ and $\{\phi_j'(b;b,\lambda_1,\mu)-\phi_j'(b;b,\lambda_2,\mu)\}=0$, $\{\phi_l'(a;a,\lambda_1,\mu)-\phi_l'(a;a,\lambda_2,\mu)\}=0$. Thus

$$[\phi_l(a;x,\lambda_1,\mu),\{\phi_j(b;x,\lambda_1,\mu)-\phi_j(b;x,\lambda_2,\mu)\}](b)=0,$$

$$[\{\phi_l(a;x,\lambda_1,\mu)-\phi_l(a;x,\lambda_2,\mu)\},\phi_j(b;x,\lambda_2,\mu)](a)=0,$$

independently from (λ_1,μ) and (λ_2,μ) respectively. Thus we can verify following:

$$\langle\phi_l(a;x,\lambda_1,\mu),L\phi_j(b;x,\lambda_2,\mu)\rangle-\langle\phi_j(b;x,\lambda_2,\mu),\phi_l(a;x,\lambda_1,\mu)\rangle=P_{lj}(\lambda_1,\mu)-P_{lj}(\lambda_2,\mu).$$

On the other hand we have

$$\langle \phi_l(a;x,\lambda_1,\mu), L\phi_j(b;x,\lambda_2,\mu)\rangle - \langle \phi_j(b;x,\lambda_2,\mu), L\phi_l(a;x,\lambda_1,\mu)\rangle$$

$$=(\lambda_2-\lambda_1)\int_a^b u_l(a;x,\lambda_1,\mu)\cdot u_j(b;x,\lambda_2,\mu)dx.$$

When $\lambda_1,\lambda_2 \Rightarrow \lambda$,we have the following relation:

(b) $$\int_a^b u_l(a;x,\lambda,\mu)\cdot u_j(b;x,\lambda,\mu)dx = -\left(\partial/\partial\lambda\right)P_{lj}(\lambda,\mu), \quad (3.2)$$

$$\int_a^b v_l(a;x,\lambda,\mu)\cdot v_j(b;x,\lambda,\mu)dx = -\left(\partial/\partial\mu\right)P_{lj}(\lambda,\mu). \quad (3.2')$$

4.Eigen values and eigenfunctions of differential operator L

In this chapter we consider the eigenfunctions corresponding to the eigenvalues of differential operator L. Take four boundary condition vectors as following:

$$\phi_l(a;x,\lambda,\mu) = (u_l(a;x,\lambda,\mu), v_l(a;x,\lambda,\mu))\ (l=1,2),$$

$$\phi_j(b;x,\lambda,\mu) = (u_j(b;x,\lambda,\mu), v_j(b;x,\lambda,\mu))\ (j=3,4).$$

We make 'Wronskian' with $\phi_l(a;x,\lambda,\mu)\ (l=1,2)$, $\phi_j(b;x,\lambda,\mu)\ (j=3,4)$:

$$W(\phi_1,\phi_2,\phi_3,\phi_4)=\begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1' & v_2' & v_3' & v_4' \\ v_1 & v_2 & v_3 & v_4 \\ u_1' & u_2' & u_3' & u_4' \end{vmatrix}.$$

Differentiating $W(\phi_1,\phi_2,\phi_3,\phi_4)$ by x, then we have $W_x(\phi_1,\phi_2,\phi_3,\phi_4)\equiv 0$. Thus $W(\phi_1,\phi_2,\phi_3,\phi_4)$ dose not depend on x but depend on λ and μ. We denote it by $W(\lambda,\mu)$. By some calculation we have following equality:

$$W(\lambda,\mu)=P_{12}P_{34}-P_{13}P_{24}+P_{14}P_{23}=P_{14}P_{23}-P_{13}P_{24}.$$

[Theorem 2] The eigenvalue $\Lambda_n(\mu)$ is a zero $(\lambda_n(\mu),\mu)$ of $W(\lambda,\mu)$. Conversely if $(\lambda_n(\mu),\mu)$ is a zero of $W(\lambda,\mu)$, there is an eigenfunction corresponding to $(\lambda_n(\mu),\mu)$.

(Proof) Let $\phi_l(a;x,\lambda_n(\mu),\mu)$ (l=1,2) and $\phi_j(b;x,\lambda_n(\mu),\mu)$ (j=3,4) be the boundary condition vectors for $(\lambda_n(\mu),\mu)$. Notice that $P_{12}(\lambda_n(\mu),\mu)=P_{34}(\lambda_n(\mu),\mu)=0$ by (2.4). For any eigenfunction $\psi(x,\lambda_n(\mu),\mu)$ we have following relation:

$$\begin{cases} [\psi(a,\lambda_n(\mu),\mu),\phi_l(a;a,\lambda_n(\mu),\mu)]=0 \\ [\psi(b,\lambda_n(\mu),\mu),\phi_j(b;b,\lambda_n(\mu),\mu)]=0. \end{cases} \tag{4.1}$$

$\psi(x,\lambda_n(\mu),\mu)$ is given by a linear combination $\psi=A\cdot\phi_1+B\cdot\phi_2+C\cdot\phi_3+D\cdot\phi_4$ why $\phi_l(a;x,\lambda_n(\mu),\mu)$ (l=1,2) and $\phi_j(b;x,\lambda_n(\mu),\mu)$ (j=3,4) form a fundamental system (A,B,C,D are constants not all zero). Take C,D, not both zero, following holds by (4.1):

$$\begin{cases} C\cdot P_{13}(\lambda_n(\mu),\mu)+D\cdot P_{14}(\lambda_n(\mu),\mu)=0, \\ C\cdot P_{23}(\lambda_n(\mu),\mu)+D\cdot P_{24}(\lambda_n(\mu),\mu)=0. \end{cases}$$

Eliminating C,D we have $P_{13}(\lambda_n(\mu),\mu)P_{24}(\lambda_n(\mu),\mu)-P_{14}(\lambda_n(\mu),\mu)P_{23}(\lambda_n(\mu),\mu)=0$. Hence $(\lambda_n(\mu),\mu)$ is a zero of $W(\lambda,\mu)$. Conversely, let $W(\lambda,\mu)=0$ at $(\lambda_n(\mu),\mu)$. Then there exist constants A,B,C,D such as, at $(\lambda_n(\mu),\mu)$,

$$\psi(x)=A\phi_1+B\phi_1=C\phi_3+D\phi_4, \tag{4.2}$$

where A,B can not both vanish why ϕ_1,ϕ_2 are independent from each other. Similarly C,D can not both vanish. Hence $\psi(x,\lambda_n(\mu),\mu)$ is not a trivial solution of given differential equation (1.2). It follows from (4.2) that ,at $(\lambda_n(\mu),\mu)$,

$$[\psi,\phi_l](a)=0\ (l=1,2),[\psi,\phi_j](b)=0\ (j=3,4). \tag{4.3}$$

By (4.3) $\psi(x)$ is an eigenfunction for the eigenvalue $\Lambda_n(\mu)(=(\lambda_n(\mu),\mu))$.□

We express it by $\psi(x)=\psi(x,\lambda_n(\mu),\mu)(=\psi(x,\Lambda_n(\mu)))$ and so on.

[Theorem 3] The eigenvalues $\Lambda_n(\mu)$, *i.e* $(\lambda_n(\mu),\mu)$, *the root of* $W(\lambda,\mu)=0$ *are all real.*

(Proof) This follows in usual manner. □

We study the zeros of $W(\lambda,\mu)$ at value $(\lambda_n(\mu),\mu)(=\Lambda_n(\mu))$. We consider the eigenvalue $(\lambda_n(\mu_0),\mu_0)$ for fixed μ_0. We express $\lambda_n(\mu_0)$ by λ_0 for simplicity for the moment. Two cases arise:

Case(i) $W(\lambda_0,\mu_0)=0$, but at least one $P_{lj}\neq 0$ at (λ_0,μ_0),

Case(ii) $W(\lambda_0,\mu_0)=0$ and all $P_{lj}=0$ at (λ_0,μ_0).

Case(i). At least one $P_{lj}\neq 0$, say $P_{23}\neq 0$, on (λ_0,μ_0). Since $W(\lambda,\mu)=0$ at (λ_0,μ_0), it follows that there is an eigenfunction

$$\psi(x,\lambda_0,\mu_0)=A\cdot\phi_1+B\cdot\phi_2=C\cdot\phi_3+D\cdot\phi_4, \tag{4.4}$$

where both A,B and both C,D can not vanish. By use of (4.3) we have

$$A\cdot P_{13}+B\cdot P_{23}=0=C\cdot P_{23}+D\cdot P_{24}. \tag{4.5}$$

Since $P_{23}\neq 0$, if $A=0$ then $B=0$. Thus $A\neq 0$. Similarly $D\neq 0$. Substituting B and C in (4.4) by (4.5) gives, at (λ_0,μ_0),

$$A\cdot(P_{23}\phi_1-P_{13}\phi_2)=D\cdot(P_{23}\phi_4-P_{24}\phi_3). \tag{4.6}$$

(4.6) is expressed by use of components of ϕ_j (j=1,2,3,4) as following:

$$\begin{cases} A\cdot(P_{23}u_1-P_{13}u_2)=D\cdot(P_{23}u_4-P_{24}u_3) \\ A\cdot(P_{23}v_1-P_{13}v_2)=D\cdot(P_{23}v_4-P_{24}v_3) \end{cases} \tag{4.6'}$$

ϕ_1,ϕ_2 are linearly independent from each other, so that $P_{23}\phi_1-P_{13}\phi_2\neq 0$ for some $x(a\le x\le b)$ and $\|P_{23}\phi_1-P_{13}\phi_2\|\neq 0$. Thus we obtain following:

$$\|P_{23}u_1-P_{13}u_2\|\neq 0 \quad \text{or} \quad \|P_{23}v_1-P_{13}v_2\|\neq 0. \tag{4.7}$$

By (4.6'), (4.7), (3.2) and (3.2'), at least (4.8) or (4.8') is verified:

$$0 \neq \left(\frac{A}{D}\right)\|P_{23}u_1 - P_{13}u_2\| = -P_{23}(\lambda_0, \mu_0)W_\lambda(\lambda_0, \mu_0), \tag{4.8}$$

$$0 \neq \left(\frac{A}{D}\right)\|P_{23}v_1 - P_{13}v_2\| = -P_{23}(\lambda_0, \mu_0)W_\mu(\lambda_0, \mu_0). \tag{4.8'}$$

Hence by (4.8) and (4.8'), $(\lambda_n(\mu_0), \mu_0)$ or $(\lambda_0, \mu_n(\lambda_0))$ is a simple zero of $W(\lambda, \mu_0)$ in the Case (i). In general, $(\lambda_n(\mu), \mu)$ that at least one $P_{lj}(\lambda_n(\mu), \mu)$ dose not vanish is a simple zero of $W(\lambda, \mu)$. Thus the eigenfunction corresponding to (λ_0, μ_0) is given by constant multiple of $P_{23}(\lambda_0, \mu_0)\phi_1(x; \lambda_0, \mu_0) - P_{13}(\lambda_0, \mu_0)\phi_2(x; \lambda_0, \mu_0)$ or $P_{23}(\lambda_0, \mu_0)\phi_4(x; \lambda_0, \mu_0) - P_{24}(\lambda_0, \mu_0)\phi_3(x; \lambda_0, \mu_0)$. The normalized eigenfunction is

$$\psi(x, \lambda_0, \mu_0) = \left\{ \frac{k_n}{\{P_{23}(\lambda_0, \mu_0)W_\lambda(\lambda_0, \mu_0)\}^{1/2}} \right\} \{P_{23}\phi_1 - P_{13}\phi_2\},$$

where $\phi_1 = \phi_1(a; x, \lambda_0, \mu_0)$, $\phi_2 = \phi_2(a; x, \lambda_0, \mu_0)$ and k_n is a constant.

[Theorem 4] We suppose $W(\lambda_0, \mu_0) = 0$ and at least one $P_{lj} \neq 0$ at (λ_0, μ_0). Then there exists one eigenvalue $(\lambda, \mu) = (\lambda_n(\mu), \mu)$ in the sufficiently small neighborhood V of μ_0. And there exists one eigenfinction $\psi(x, \lambda_n(\mu), \mu)$ corresponding to $(\lambda, \mu) = (\lambda_n(\mu), \mu)$ in V.

(Proof) Since $W(\lambda, \mu)$ is a regular analytic function in the small neighborhood of (λ_0, μ_0) and (λ_0, μ_0) is a simple zero of $W(\lambda, \mu)$, for each $\mu \in V$ (V is a small neighborhood of μ_0), $W(\lambda, \mu)$ has a simple zero $(\lambda, \mu) = (\lambda_n(\mu), \mu)$. By [Theorem 3] there is an eigenfunction $\psi(x, \lambda_n(\mu), \mu)$ for $(\lambda, \mu) = (\lambda_n(\mu), \mu)$. □

Case (ii). In this case $P_{lj}(\lambda_0, \mu_0) = 0$ (l =1,2, j =3,4) we have following expression:

$$W(\lambda_0, \mu_0) = P_{14}(\lambda_0, \mu_0)P_{23}(\lambda_0, \mu_0) - P_{24}(\lambda_0, \mu_0)P_{13}(\lambda_0, \mu_0) = 0,$$

$$W_\lambda(\lambda_0, \mu_0) = P_{14\lambda}(\lambda_0, \mu_0)P_{23}(\lambda_0, \mu_0) + P_{14}(\lambda_0, \mu_0)P_{23\lambda}(\lambda_0, \mu_0),$$
$$- P_{24\lambda}(\lambda_0, \mu_0)P_{13}(\lambda_0, \mu_0) - P_{24}(\lambda_0, \mu_0)P_{13\lambda}(\lambda_0, \mu_0) = 0,$$

$$W_\mu(\lambda_0, \mu_0) = P_{14\mu}(\lambda_0, \mu_0)P_{23}(\lambda_0, \mu_0) + P_{14}(\lambda_0, \mu_0)P_{23\mu}(\lambda_0, \mu_0)$$
$$- P_{24\mu}(\lambda_0, \mu_0)P_{13}(\lambda_0, \mu_0) - P_{24}(\lambda_0, \mu_0)P_{13\mu}(\lambda_0, \mu_0) = 0$$

$$W_{\lambda\lambda}(\lambda_0, \mu_0) = 2\{P_{14\lambda}(\lambda_0, \mu_0)P_{23\lambda}(\lambda_0, \mu_0) - P_{24\lambda}(\lambda_0, \mu_0)P_{13\lambda}(\lambda_0, \mu_0)\},$$

$$W_{\mu\mu}(\lambda_0, \mu_0) = 2\{P_{14\mu}(\lambda_0, \mu_0)P_{23\mu}(\lambda_0, \mu_0) - P_{24\mu}(\lambda_0, \mu_0)P_{13\mu}(\lambda_0, \mu_0)\},$$

$$W_{\lambda\mu}(\lambda_0,\mu_0)=P_{14\mu}(\lambda_0,\mu_0)P_{23\lambda}(\lambda_0,\mu_0)+P_{14\lambda}(\lambda_0,\mu_0)P_{23\mu}(\lambda_0,\mu_0)$$
$$-P_{24\mu}(\lambda_0,\mu_0)P_{13\lambda}(\lambda_0,\mu_0)-P_{24\lambda}(\lambda_0,\mu_0)P_{13\mu}(\lambda_0,\mu_0).$$

[Theorem 5] Let ϕ_3,ϕ_4 be linearly dependent on ϕ_1,ϕ_2 at (λ_0,μ_0), then P_{13},P_{14},P_{23} and P_{24} vanish there. Conversely P_{13},P_{14}, P_{23} and P_{24} vanish at (λ_0,μ_0), then ϕ_3,ϕ_4 are linearly dependent on ϕ_1,ϕ_2 at (λ_0,μ_0).

(Proof) We suppose that ϕ_3,ϕ_4 be linearly dependent on ϕ_1,ϕ_2 at (λ_0,μ_0). Then

$$\phi_3(b;x,\lambda_0,\mu_0)=A\cdot\phi_1(a;x,\lambda_0,\mu_0)+B\cdot\phi_2(a;x,\lambda_0,\mu_0)$$
$$\phi_4(b;x,\lambda_0,\mu_0)=C\cdot\phi_1(a;x,\lambda_0,\mu_0)+D\cdot\phi_2(a;x,\lambda_0,\mu_0),$$

where A,B,C,D are some constants. By use of above equalities we have $P_{13}=P_{14}=P_{23}=P_{24}=0$ at (λ_0,μ_0),why $[\phi_1,\phi_2]=0,[\phi_1,\phi_1]=0,[\phi_2,\phi_2]=0$ by use of the properties of bilinear forms. Conversely we suppose $P_{13}=P_{14}=P_{23}=P_{24}=0$ at (λ_0,μ_0). Put $\phi_j=(u_j,v_j)(j=1,2,3,4)$ then we have for every minor of order three of $W(\lambda_0,\mu_0)$ as following at (λ_0,μ_0), say:

$$\begin{vmatrix} u_1' & u_2' & u_3' \\ u_1 & u_2 & u_3 \\ v_1' & v_2' & v_3' \end{vmatrix}=u_1'P_{23}-u_2'P_{13}+u_3'P_{12}=0.$$

The same results hold for other minor of order three. Thus $W(\lambda_0,\mu_0)=0$ and ϕ_3,ϕ_4 depend on ϕ_1,ϕ_2 at (λ_0,μ_0). Theorem is proved□

By this theorem, at (λ_0,μ_0), we have the expression:

$$\begin{cases} \phi_3(b;x,\lambda_0,\mu_0)=A\cdot\phi_1(b;x,\lambda_0,\mu_0)+B\cdot\phi_2(a;x,\lambda_0,\mu_0) \\ \phi_4(b;x,\lambda_0,\mu_0)=C\cdot\phi_1(b;x,\lambda_0,\mu_0)+D\cdot\phi_2(a;x,\lambda_0,\mu_0), \end{cases} \tag{4.9}$$

where $\Delta=A\cdot D-B\cdot C\neq 0$. Put

$$I_{ij}=\int_a^b u_i(a;x,\lambda_0,\mu_0)u_j(a;x,\lambda_0,\mu_0)dx\ ,J_{ij}=\int_a^b v_i(a;x,\lambda_0,\mu_0)v_j(a;x,\lambda_0,\mu_0)dx$$

then we have $I_{ij}=I_{ji},J_{ij}=J_{ji}\ (i,j=1,2)$. By (3.2) ,(3.2') following hold at (λ_0,μ_0):

$$\begin{cases} -P_{13\lambda} = A\cdot I_{11} + B\cdot I_{12}, -P_{14\lambda} = C\cdot I_{11} + D\cdot I_{12} \\ -P_{23\lambda} = A\cdot I_{21} + B\cdot I_{22}, -P_{24\lambda} = C\cdot I_{21} + D\cdot I_{22} \end{cases} \tag{4.10}$$

$$\begin{cases} -P_{13\mu} = A\cdot J_{11} + B\cdot J_{12}, -P_{14\mu} = C\cdot J_{11} + D\cdot J_{12}, \\ -P_{23\mu} = A\cdot J_{21} + B\cdot J_{22}, -P_{24\mu} = C\cdot J_{21} + D\cdot J_{22} \end{cases} \tag{4.10'}$$

By the condition $W_{\lambda\lambda}(\lambda_0,\mu_0) = 2\Delta(I_{12} - I_{11}\cdot I_{22}), W_{\mu\mu}(\lambda_0,\mu_0) = 2\Delta(J_{12} - J_{11}\cdot J_{22})$ and $W_{\lambda\mu}(\lambda_0,\mu_0) = \Delta(2I_{12}\cdot J_{12} - I_{11}\cdot J_{22} - I_{22}\cdot J_{11})$. By the definition of I_{ij}, J_{ij} and (4.10), (4.10') following inequalities hold:

$$I_{12}^{\ 2} \le \left\{\int_a^b |u_1 u_2| dx\right\}^2 \le \left(\int_a^b u_1^{\ 2} dx\right)\left(\int_a^b u_2^{\ 2} dx\right) \le I_{11}\cdot I_{22},$$

$$J_{12}^{\ 2} \le \left\{\int_a^b |v_1 v_2| dx\right\}^2 \le \left(\int_a^b v_1^{\ 2} dx\right)\left(\int_a^b v_2^{\ 2} dx\right) \le J_{11}\cdot J_{22}.$$

If $I_{12}^{\ 2} = I_{11}\cdot I_{22}$ and $J_{12}^{\ 2} = J_{11}\cdot J_{22}$ then ϕ_1 must be linearly dependent on ϕ_2. This contradicts our assumption. Thus $I_{12}^{\ 2} < I_{11}\cdot I_{22}$ or $J_{12}^{\ 2} < J_{11}\cdot J_{22}$. Thus we can prove $W_{\lambda\lambda}(\lambda_0,\mu_0) \ne 0$ or $W_{\mu\mu}(\lambda_0,\mu_0) \ne 0$. This means that (λ_0,μ_0) is a double zero of $W(\lambda,\mu)$ why $W(\lambda_0,\mu_0) = 0, W_\lambda(\lambda_0,\mu_0) = 0, W_\mu(\lambda_0,\mu_0) = 0$.

$\phi_3(b;x,\lambda_0,\mu_0)$ and $\phi_4(b;x,\lambda_0,\mu_0)$ satisfy the boundary condition at $x = a, x = b$. Thus the normalized eigenfunctions $\psi_1(x,\lambda_0,\mu_0)$ and $\psi_2(x,\lambda_0,\mu_0)$ are expressed by linear combination of $\phi_3(b;x,\lambda_0,\mu_0)$, $\phi_4(b;x,\lambda_0,\mu_0)$,in other words, these normalized eigenfunctions are expressed by linear combination of ϕ_1, ϕ_2:

$$\psi_1 = (A\phi_1 + B\phi_2)\Big/\sqrt{\langle\phi_3,\phi_3\rangle}$$

$$\psi_2 = \gamma\left\{(A\phi_1 + B\phi_2)\Big/\sqrt{\langle\phi_3,\phi_3\rangle} - \sqrt{\langle\phi_3,\phi_3\rangle}(C\phi_1 + D\phi_2)/\langle\phi_3,\phi_4\rangle\right\},$$

where $\langle\phi_3,\phi_3\rangle = A^2\langle\phi_1,\phi_1\rangle + 2AB\langle\phi_1,\phi_2\rangle + B^2\langle\phi_2,\phi_2\rangle$,

$\langle\phi_4,\phi_4\rangle = C^2\langle\phi_1,\phi_1\rangle + 2CD\langle\phi_1,\phi_2\rangle + D^2\langle\phi_2,\phi_2\rangle$,

$\langle\phi_3,\phi_4\rangle = AC\ \langle\phi_1,\phi_1\rangle + (BC + AD)\langle\phi_1,\phi_2\rangle + BD\langle\phi_2,\phi_2\rangle$,

$$\gamma = \left\{ \langle \phi_3, \phi_4 \rangle \Big/ (AD - BC) \right\} \left\{ 1 \Big/ \sqrt{\langle \phi_1, \phi_1 \rangle \langle \phi_2, \phi_2 \rangle - \langle \phi_1, \phi_2 \rangle^2} \right\}.$$

Hence we obtain following theorem.

[Theorem 6] If $W(\lambda_0, \mu_0) = 0$ and all $P_{lj} = 0$ at (λ_0, μ_0), there is a double eigenvalue $(\lambda_0, \mu_0) = (\lambda_n(\mu_0), \mu_0)$ and two normalized eigenfunctions $\psi_1(x, \lambda_0, \mu_0)$ and $\psi_2(x, \lambda_0, \mu_0)$ corresponding to $(\lambda_0, \mu_0) = (\lambda_n(\mu_0), \mu_0)$.

(Proof) By above discussion, the existence of normalized eigenfunctions $\psi_1(x, \lambda_0, \mu_0)$ and $\psi_2(x, \lambda_0, \mu_0)$ corresponding to (λ_0, μ_0) is verified.□

(Example 1) Consider the boundary value problem:

$$\begin{cases} v'' = \lambda \cdot u, \\ u'' = \mu \cdot v, \end{cases}$$

$$v'(0) = 0, u'(0) = 0,$$

$$v'(1) = 0, u'(1) = 0.$$

This problem has been studied by Chakravarty[3] when $\lambda = \mu$. We take vectors $\phi_1(0; x, \lambda, \mu) = (u_l(0; x, \lambda, \mu), v_l(0; x, \lambda, \mu)), \phi_j(1; x, \lambda, \mu) = (u_j(1; x, \lambda, \mu), v_j(1; x, \lambda, \mu))$ $(l = 1,2; j = 3,4)$. For example we take $\phi_1, \phi_2, \phi_3, \phi_4$ under the following condition:

$$u_1(0) = 1, u_1'(0) = 0, v_1(0) = 0, v_1'(0) = 0,$$

$$u_2(0) = 0, u_2'(0) = 0, v_2(0) = 1, v_2'(0) = 0,$$

$$u_3(1) = 1, u_3'(1) = 0, v_3(1) = 0, v_3'(1) = 0,$$

$$u_4(1) = 0, u_4'(1) = 0, v_4(1) = 1, v_4'(1) = 0.$$

About these boundary condition vectors we verify some identities by use of Mathematica:

$$\int_0^1 u_1(0, x, \lambda, \mu) \cdot u_3(1; x, \lambda, \mu) dx \equiv$$

$$\{\lambda\mu(\cos[(\lambda\mu)^{1/4}] + \cosh[(\lambda\mu)^{1/4}]) + 3(\lambda\mu)^{1/4}(\sin[(\lambda\mu)^{1/4}] + \sinh[(\lambda\mu)^{1/4}])\} / 8\lambda\mu,$$

$$\int_0^1 v_1(0;x,\lambda,\mu)\cdot v_3(1;x,\lambda,\mu)dx \equiv$$

$$\{\lambda\mu(\cos[(\lambda\mu)^{1/4}]+\cosh[(\lambda\mu)^{1/4}])-(\lambda\mu)^{3/4}(\sin[(\lambda\mu)^{1/4}]+\sinh[(\lambda\mu)^{1/4}])\}/8\mu^2.$$

On the other hand we obtain following identities:

$$P_{13}(\lambda,\mu)\equiv\{-(\lambda\mu)^{1/4}\}(\sin[(\lambda\mu)^{1/4}]+\sinh[(\lambda\mu)^{1/4}])/2(\mu/\lambda)^{1/2},$$

$$P_{13\lambda}(\lambda,\mu)\equiv-\{(\cos[(\lambda\mu)^{1/4}]+\cosh[(\lambda\mu)^{1/4}])+3(\lambda\mu)^{-3/4}(\sin[(\lambda\mu)^{1/4}]+\sinh[(\lambda\mu)^{1/4}])\}/8,$$

$$P_{13\mu}(\lambda,\mu)\equiv-\{(\lambda/\mu)(\cos[(\lambda\mu)^{1/4}]+\cosh[(\lambda\mu)^{1/4}])-\lambda^{3/4}\mu^{-5/4}(\sin[(\lambda\mu)^{1/4}]+\sinh[(\lambda\mu)^{1/4}])\}/8.$$

Thus we can show

$$\int_0^1 u_1(0;x,\lambda,\mu)\cdot u_3(1;x,\lambda,\mu)dx\equiv-P_{13\lambda}(\lambda,\mu),$$

$$\int_0^1 v_1(0;x,\lambda,\mu)\cdot v_3(1;x,\lambda,\mu)dx\equiv-P_{13\mu}(\lambda,\mu)$$

And the Wronskian for this problem is given by following:

$$W(\lambda,\mu)=-(\lambda\mu)^{1/2}\sin[(\lambda\mu)^{1/4}]\sinh[(\lambda\mu)^{1/4}].$$

$W(\lambda,\mu)$ is an integral function of λ,μ and it's zeros are $(\lambda,\mu)=(\lambda(\mu),\mu)=(n^4\pi^4/\mu,\mu)$ $(n\in N)$ and they are simple eigenvalues for this problem. They are given by curves in the $\lambda\cdot\mu$ plane. The eigenfunctions for these eigenvalues are obtained as following:

$$\psi(x,(n\pi)^4/\mu,\mu)=0.5\cdot n\pi\cdot((n\pi)^2/\mu,\quad -1)^T\cos[(n\pi)x]\cdot\sinh[(n\pi)].$$

$\psi(x,(n\pi)^4/\mu,\mu)$ has n zeros in the interval $[0,1]$.

(Example 2) Consider the boundary value problem:

$$\begin{cases} v'' = \lambda\cdot u, \\ u'' = \mu\cdot v, \end{cases}$$

$$4\cdot v(0)-v'(0)=0, 4\cdot u(0)-u'(0)=0,$$

$$4\cdot v(1)-v'(1)=0, 4\cdot u(1)-u'(1)=0.$$

For this boundary value problem we determine the boundary condition vectors

$$\phi_1(0;x,\lambda,\mu)=(u_I(0;x,\lambda,\mu),v_I(0;x,\lambda,\mu)),\phi_j(1;x,\lambda,\mu)=(u_j(1;x,\lambda,\mu),v_j(1;x,\lambda,\mu))$$

$(l = 1,2; j = 3,4)$ as the solutions of above equation under the following condition:

$$u_1(0)=1, v_1(0)=0, u_1'(0)=4, v_1'(0)=0,$$
$$u_2(0)=0, v_2(0)=1, u_2'(0)=0, v_2'(0)=4,$$
$$u_3(0)=1, v_3(0)=0, u_3'(0)=4, v_3'(0)=0,$$
$$u_4(0)=0, v_4(0)=1, u_4'(0)=0, v_4'(0)=4.$$

For these boundary condition vectors we obtained the following identities by use of Mathematica:

$$W(\lambda,\mu)=0.5\cdot\{\exp((\lambda\cdot\mu)^{1/4})-\exp(-(\lambda\cdot\mu)^{1/4})\}(256-\lambda\cdot\mu)(\lambda\cdot\mu)^{-1/2}\sin((\lambda\cdot\mu)^{1/4})$$

This $W(\lambda,\mu)$ is an integral function of λ,μ. The eigenvalues are expressed by $(\lambda_0(\mu),\mu)=\left(256/\mu,\mu\right)$ and $(\lambda_n(\mu),\mu)=\left((n\pi)^4/\mu,\mu\right)\ (n\in N)$. These eigenvalues are all simple zeros of $W(\lambda,\mu)$. Eigenfunctions are given as following:

$$\psi(x,256/\mu,\mu)=e^{4x}\sin[4]\,(64/\mu,4)^T,$$

$$\psi(x,(n\pi)^4/\mu,\mu)=(e^{n\pi}-e^{-n\pi})(n^2\pi^2-16)\sqrt{(16+n^2\pi^2)}\sin[n\pi x+\alpha](1/4\mu,-1/n^2 4\pi^2)^T,$$

$\tan[\alpha]=n\pi/4$.

$\psi(x,256/\mu,\mu)$ dose not vanish in the interval [0,1] and $\psi(x,(n\pi)^4/\mu,\mu)$ has n zeros in the interval [0,1].

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