Integrability of Dynamical Systems and Gauss Hypergeometric Equation

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Abstract

In Ishii [1] and [2], we considered the Kowalevskii's exponent of some Hamiltonian systems with a parameter e. And by use of the Kowalevskii's exponent we considered the integrability of those Hamiltonian systems (see Yoshida [5], [7]). We could determine the values of e which are possible to give the integrability of those Hamiltonian sytems. But we dont know that those Hamiltonian systems are integrable or not for these values of e. We studied the integrability of that Hamiltonian systems more closely by use of Morales-Luiz, Ramis [4] and Kimura [3].

Introduction

We cosidered the Kowalevskii's exponent of some nonlinear dynamics (see [1] and [2]). In these papers we studied the integrability of nonlinear dynamics by use of Kowalevskii's exponents.

In this paper we consider the integrability of dynamics from another points of view. We considered the Hamiltonian system :

$$H(q_1,\ q_2,\ p_1,\ p_2) = \frac{1}{2}(p_1^2 + p_2^2) + \epsilon(q_1^4 + q_2^4) + q_1^2 q_2^2.$$

These Hamilltonian systems are integrable for only $\epsilon = 1/2$, $1/6$ (see [1]). When $\epsilon = 1/2$, the differential equations derived from above Hamiltonian are written as following by scale change $q_i = \frac{1}{\sqrt{2}} Q_i$, $p_i = \frac{1}{\sqrt{2}} P_i$ $(i = 1, 2)$:

$$\frac{d}{dt} Q_1 = P_1,$$

$$\frac{d}{dt} Q_2 = P_2,$$

$$\frac{d}{dt} P_1 = -Q_1^3 - Q_1 Q_2^2,$$

$$\frac{d}{dt} P_2 = -Q_2^3 - Q_1^2 Q_2.$$

When $\epsilon = 1/6$, the differential equations derived from above Hamiltonian are written as following by scale change $q_i=\sqrt{\frac{3}{2}}Q_i$, $p_i=\sqrt{\frac{3}{2}}P_i$ $(i=1, 2)$:

$$\frac{d}{dt}Q_1=P_1,$$
$$\frac{d}{dt}Q_2=P_2,$$
$$\frac{d}{dt}P_1=-Q_1^3-3Q_1Q_2^2,$$
$$\frac{d}{dt}P_2=-Q_2^3-3Q_1^2Q_2.$$

These systems of differential equations are derived from an integrable Hamiltonian

$$H^*(Q_1, Q_2, P_1, P_2)=\frac{1}{2}(P_1^2+P_2^2)+\frac{1}{4}(Q_1^4+Q_2^4)+\epsilon Q_1^2Q_2^2$$

by taking $\epsilon=1/2, 3/2$.

In Ishii [2] by Kowalevskii's exponents we considered the integrability of perturved Hamiltonian system (1). And we showed that (1) has rational Kowalevskii's exponents for $\epsilon=3/2$ independent from e, and for $\epsilon=1/2$, (1) has rational Kowalevskii's exponents when there is a rational solution of the indeterminate equation

$$20x^2+20y^2-16x^2y^2=9$$

and for the other complex e there dose not exist any rational Kowalevskii's exponent.

Therfore, in this paper we study the perturved differential equations derived from

$$H(q_1, q_2, p_1, p_2 ; e)=\frac{1}{2}(p_1^2+p_2^2)+\frac{1}{4}(q_1^4+q_2^4)+e(q_1^3q_2+q_1q_2^3)+\epsilon q_1^2q_2^2 \quad (e\neq 0, \infty) \tag{1}$$

at $\epsilon=1/2, 3/2$ from another point of view. Following differential equations are considered:

$$\frac{d}{dt}q_1=\frac{\partial}{\partial p_1}H(q_1, q_2, p_1, p_2 ; e) \tag{2}$$
$$\frac{d}{dt}q_2=\frac{\partial}{\partial p_2}H(q_1, q_2, p_1, p_2 ; e) \tag{3}$$
$$\frac{d}{dt}p_1=-\frac{\partial}{\partial q_1}H(q_1, q_2, p_1, p_2 ; e) \tag{4}$$
$$\frac{d}{dt}p_2=-\frac{\partial}{\partial q_2}H(q_1, q_2, p_1, p_2 ; e). \tag{5}$$

1. The straight-line solutions

The straight-line solution (see Yoshida [5]) is $(q_1, q_2)=(c_1, c_2)Q(t)$, $(p_1, p_2)=(c_1, c_2)P(t)$, where (c_1, c_2) is the solution of

$$c_1=H_{q_1}(c_1, c_2, c_1, c_2 ; e),\ c_2=H_{q_2}(c_1, c_2, c_1, c_2 ; e), \tag{6}$$

and $Q(t)$, $P(t)$ is given as the solution of

$$\frac{d}{dt}Q=P, \quad \frac{d}{dt}P=-Q^3. \tag{7}$$

The equation (7) is derived by the one-degree-of-freedom Hamiltonian

$$h=\frac{1}{2}P^2+\frac{1}{4}Q^4.$$

If we set the value of energy, $h=1/4$, we find that $Q=Q(t)$ is obtained by the inverse function of

$$t-t_0=\int\left[\frac{1}{2}\left(1-Q^4\right)\right]^{-1/2}dQ. \tag{8}$$

Thus we have $P=P(t)$ by use of $Q=Q(t)$. We consider equations (6).

Case $\epsilon=3/2$.

The solutions of (6) are given as following :

$$(c_1^{(1)},\ c_2^{(1)})=\left(\frac{-1}{2}\left(\frac{1}{\sqrt{1-e}}+\frac{1}{\sqrt{1+e}}\right),\ \frac{1}{2}\left(\frac{1}{\sqrt{1-e}}-\frac{1}{\sqrt{1+e}}\right)\right),$$

$$(c_1^{(2)},\ c_2^{(2)})=\left(\frac{1}{2}\left(\frac{-1}{\sqrt{1-e}}+\frac{1}{\sqrt{1+e}}\right),\ \frac{1}{2}\left(\frac{1}{\sqrt{1-e}}+\frac{1}{\sqrt{1+e}}\right)\right),$$

$$(c_1^{(3)},\ c_2^{(3)})=\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-e}}-\frac{1}{\sqrt{1+e}}\right),\ \frac{-1}{2}\left(\frac{1}{\sqrt{1-e}}+\frac{1}{\sqrt{1+e}}\right)\right),$$

$$(c_1^{(4)},\ c_2^{(4)})=\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-e}}+\frac{1}{\sqrt{1+e}}\right),\ \frac{1}{2}\left(\frac{-1}{\sqrt{1-e}}+\frac{1}{\sqrt{1+e}}\right)\right),$$

$$(c_1^{(5)},\ c_2^{(5)})=\left(\frac{-1}{2\sqrt{1+e}},\ \frac{-1}{2\sqrt{1+e}}\right),\ (c_1^{(6)},\ c_2^{(6)})=\left(\frac{1}{2\sqrt{1+e}},\ \frac{1}{2\sqrt{1+e}}\right),$$

$$(c_1^{(7)},\ c_2^{(7)})=\left(\frac{-1}{2\sqrt{1-e}},\ \frac{1}{2\sqrt{1-e}}\right),\ (c_1^{(8)},\ c_2^{(8)})=\left(\frac{1}{2\sqrt{1-e}},\ \frac{-1}{2\sqrt{1-e}}\right),$$

$$(c_1^{(9)},\ c_2^{(9)})=(0,\ 0).$$

Case $\epsilon=1/2$.

The solutions of (6) are given as following :

$$(c_1^{(1)},\ c_2^{(1)})=\left(\frac{(-1)^{3/4}}{\sqrt{2}\sqrt{e}},\ -\frac{(-1)^{1/4}}{\sqrt{2}\sqrt{e}}\right),\ (c_1^{(2)},\ c_2^{(2)})=\left(-\frac{(-1)^{3/4}}{\sqrt{2}\sqrt{e}},\ \frac{(-1)^{1/4}}{\sqrt{2}\sqrt{e}}\right),$$

$$(c_1^{(3)},\ c_2^{(3)})=\left(\frac{(-1)^{1/4}}{\sqrt{2}\sqrt{e}},\ -\frac{(-1)^{3/4}}{\sqrt{2}\sqrt{e}}\right),\ (c_1^{(4)},\ c_2^{(4)})=\left(-\frac{(-1)^{1/4}}{\sqrt{2}\sqrt{e}},\ \frac{(-1)^{3/4}}{\sqrt{2}\sqrt{e}}\right),$$

$$(c_1^{(5)},\ c_2^{(5)})=\left(-\frac{1}{\sqrt{2+4e}},\ -\frac{1}{\sqrt{2+4e}}\right),\ (c_1^{(6)},\ c_2^{(6)})=\left(\frac{1}{\sqrt{2+4e}},\ \frac{1}{\sqrt{2+4e}}\right),$$

$$(c_1^{(7)},\ c_2^{(7)})=\left(\frac{1}{\sqrt{2-4e}},\ -\frac{1}{\sqrt{2-4e}}\right),\ (c_1^{(8)},\ c_2^{(8)})=\left(-\frac{1}{\sqrt{2-4e}},\ \frac{1}{\sqrt{2-4e}}\right),$$

$$(c_1^{(9)},\ c_2^{(9)})=(0,\ 0).$$

2. Variational equations and Ramani's theorem

With the definition of $(\xi_1, \xi_2)^T = (\delta q_1, \delta q_2)^T$ the variational equation around the straight-line solution is

$$\frac{d^2}{dt^2}\xi = -Q(t)^2 D^2 H(c_1, c_2)\xi, \quad \xi = (\xi_1, \xi_2)^T \tag{9}$$

where $D^2H(c_1, c_2)$ is Hessian of $H(q_1, q_2, p_1, p_2; e)$ at (c_1, c_2, c_1, c_2).

Case $\epsilon = 3/2$.

At the solution $(c_1^{(i)}, c_2^{(i)})$ $(i=1, 2, 3, 4, 5, 6, 7, 8)$ we have

$$D^2H(c_1^{(i)}, c_2^{(i)}) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \ (i=1, 2, 3, 4),$$

$$D^2H(c_1^{(i)}, c_2^{(i)}) = \begin{pmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{pmatrix} \ (i=5, 6),$$

$$D^2H(c_1^{(i)}, c_2^{(i)}) = \begin{pmatrix} 3/2 & -3/2 \\ -3/2 & 3/2 \end{pmatrix} \ (i=7, 8).$$

By use of a linear canonical change of variables $\xi = S\xi'$ with an appropriate matrix S, we have variational equations

$$\frac{d^2}{dt^2}\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = -Q(t)^2 \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} \ (i=1, 2, 3, 4), \tag{10}$$

$$\frac{d^2}{dt^2}\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = -Q(t)^2 \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} \ (i=5, 6, 7, 8). \tag{11}$$

Case $\epsilon = 1/2$.

At the solution $(c_1^{(i)}, c_2^{(i)})$ $(i=1, 2, 3, 4, 5, 6, 7, 8)$ we have

$$D^2H(c_1^{(i)}, c_2^{(i)}) = \begin{pmatrix} 3-\frac{i}{e} & \frac{1}{e} \\ \frac{1}{e} & 3-\frac{i}{e} \end{pmatrix} \ (i=1, 2, 3, 4),$$

$$D^2H(c_1^{(i)}, c_2^{(i)}) = \begin{pmatrix} (2+3e)/(1+2e) & (1+3e)/(1+2e) \\ (1+3e)/(1+2e) & (2+3e)/(1+2e) \end{pmatrix} \ (i=5, 6),$$

$$D^2H(c_1^{(i)}, c_2^{(i)}) = \begin{pmatrix} (2-3e)/(1-2e) & (-1+3e)/(1-2e) \\ (-1+3e)/(1-2e) & (2-3e)/(1-2e) \end{pmatrix} \ (i=7, 8).$$

By use of a linear canonical change of variables $\xi = S\xi'$ with an appropriate matrix S, we have variational equations

$$\frac{d^2}{dt^2}\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = -Q(t)^2 \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} \ (i=1, 2, 3, 4), \tag{12}$$

$$\frac{d^2}{dt^2}\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = -Q(t)^2 \begin{pmatrix} 1/(1+2e) & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} \ (i=5, 6), \tag{13}$$

$$\frac{d^2}{dt^2}\begin{pmatrix}\xi'_1\\ \xi'_2\end{pmatrix}=-Q(t)^2\begin{pmatrix}1/(1-2e) & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}\xi'_1\\ \xi'_2\end{pmatrix}\ (i=7,\ 8), \tag{14}$$

In the later discussion we rewirite ξ' by ξ.

The normal variational equations, in above variational equations, are as following :

Case $\epsilon=3/2$.

$$\frac{d^2}{dt^2}\xi_1=-3Q(t)^2\xi_1\ \ (i=1,\ 2,\ 3,\ 4), \tag{15}$$

$$\frac{d^2}{dt^2}\xi_1=0\ \ (i=5,\ 6,\ 7,\ 8). \tag{16}$$

Case $\epsilon=1/2$.

$$\frac{d^2}{dt^2}\xi_1=-3Q(t)^2\xi_1\ \ (i=1,\ 2,\ 3,\ 4), \tag{17}$$

$$\frac{d^2}{dt^2}\xi_1=-\frac{Q(t)^2}{1+2e}\xi_1\ \ (i=5,\ 6), \tag{18}$$

$$\frac{d^2}{dt^2}\xi_1=-\frac{Q(t)^2}{1-2e}\xi_1\ \ (i=7,\ 8). \tag{19}$$

Equations (15), (16), (17), (18) can be considered as the normal variational equations around straight-line solution $p=q=0$ from the first.

Morales-Luiz, Ramis [4] showed that if a Hamilton system is integrable then the hypergeometric equation, derived from the normal variational equation around the straight-line solution $q=p=0$ of this Hamilton system, must be integrable.

So that we derive Gauss hypergeometric equation from above normal variational equation in the following section.

3. Gauss hypergeometric equation and Riemann's P-function

We derive the Gauss hypergeometric equations from these normal variational equations by change of independent variable :

$$z=[Q(t)]^4. \tag{20}$$

The Gauss hypergeometric equation corresponding to above normal variational equations is written as follwing :

$$z(1-z)\frac{d^2}{dz^2}\xi+\{c-(a+b+1)z\}\frac{d}{dz}\xi-ab\xi=0, \tag{21}$$

where parameters a,b,c are determined corresponding to some cases.

For the Riemann's P function corresponding to (21)

$$\xi(z)=P\begin{Bmatrix}0 & 1 & \infty & \\ 0 & 0 & a & z\\ 1-c & c-a-b & b & \end{Bmatrix}$$

define the difference of exponents by $\hat{\lambda}=1-c$, $\hat{\mu}=c-a-b$, $\hat{\nu}=b-a$.

Case 1: Corresponding to (15), (17), we have $c=3/4$, $a+b=1/4$, $ab=-3/8$. The hypergeometric equation is determined with $a=-1/2$, $b=3/4$, $c=3/4$. Riemann's function is given by

$$\xi(z)=P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & -1/2 & z \\ 1/4 & 1/2 & 3/4 & \end{matrix}\right\}. \tag{22}$$

In this case $\hat{\lambda}=1/4$, $\hat{\mu}=1/2$, $\hat{\nu}=5/4$.

Case 2: Corresponding to (16), we have $c=3/4$, $a+b=1/4$, $ab=0$. The hypergeometric equation is determined with $a=0$, $b=1/4$, $c=3/4$. Riemann's function is given by

$$\xi(z)=P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & 0 & z \\ 1/4 & 1/2 & 1/4 & \end{matrix}\right\}. \tag{23}$$

In this case $\hat{\lambda}=1/4$, $\hat{\mu}=1/2$, $\hat{\nu}=1/4$.

Case 3: Corresponding to (18), we have $c=\frac{3}{4}$, $a+b=\frac{1}{4}$, $ab=-\frac{1}{8}\frac{1}{1+2e}$. The hypergeometric equation is determined with $a,\ b=\left(\frac{1}{4}\pm\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}}\right)\frac{1}{2}$, $c=\frac{3}{4}$. Riemann's function is given by

$$\xi(z)=P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \left(\frac{1}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}}\right)\frac{1}{2} & z \\ \frac{1}{4} & \frac{1}{2} & \left(\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}}\right)\frac{1}{2} & \end{matrix}\right\}. \tag{24}$$

In this case $\hat{\lambda}=\frac{1}{4}$, $\hat{\mu}=\frac{1}{2}$, $\hat{\nu}=\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}}$.

Case 4: Corresponding to (19), we have $c=\frac{3}{4}$, $a+b=\frac{1}{4}$, $ab=-\frac{1}{8}\frac{1}{1-2e}$. The hypergeometric equation is determined with $a,\ b=\left(\frac{1}{4}\pm\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}}\right)\frac{1}{2}$, $c=\frac{3}{4}$. Riemann's function is given by

$$\xi(z)=P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \left(\frac{1}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}}\right)\frac{1}{2} & z \\ \frac{1}{4} & \frac{1}{2} & \left(\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}}\right)\frac{1}{2} & \end{matrix}\right\}. \tag{25}$$

In this case $\hat{\lambda}=\frac{1}{4}$, $\hat{\mu}=\frac{1}{2}$, $\hat{\nu}=\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}}$.

4. Application of T. Kimura's theorem

In this Chapter we consider the intgrability of the hypergeometric equation (E) :

$$z(1-z)\frac{d^2}{dz^2}\xi+(\gamma-(\alpha+\beta+1)z)\frac{d}{dz}\xi-\alpha\beta\xi=0. \tag{E}$$

K is the field of the set of all rational functions, the field which is obtained from K by the adjunction of the solutions of the linear ordinary differential equation considered, is called the Picard-Vessiot extension of K. The field L, which is obtained from K by a number of steps each of which is either a finite algebraic extension or the adjunction of an indefinite integral or the adjunction of an exponential of an indefinite integral, is called the generarized Liouville extension.

The solution of hypergeometric Equation (E) is written in the form of Riemann's P function as

$$\xi(z)=P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix}. \tag{26}$$

Let $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$ at $z=0, 1, \infty$ then $\hat{\lambda}=1-\gamma$, $\hat{\mu}=\gamma-\alpha-\beta$.

Kimura's theorem is stated as following (see H. Yoshida [7]) :

[**Theorem**] ([**3**], [**7**]) Let L be the Picard-Vessiot extension of K for Gauss hypergeometric equation (E). Inorder to be a generalized Liouville extension of K, it is necessary and sufficient that either

(i) at least one of $\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $-\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $\hat{\lambda}-\hat{\mu}+\hat{\nu}$, $\hat{\lambda}+\hat{\mu}-\hat{\nu}$ is an odd integer, or

(ii) $\pm\hat{\lambda}$, $\pm\hat{\mu}$, $\pm\hat{\nu}$ take values in Table, called the Table of Schwarz-Hukuhara-Ohasi in an arbitrary order, with integer l, m, n.

In following discussion the integrability of Gauss hypergeometric equation (21) is considered by use of [Theorem].

Case 1 : $c=3/4$, $a+b=1/4$, $ab=-3/8$ ($a=-1/2$, $b=3/4$, $c=3/4$).

In this case case $\hat{\lambda}-\hat{\mu}+\hat{\nu}=1$ and hypergeometric equation (21), derived from (15) or (17), is integrable with these a, b, c.

Case 2 : $c=3/4$, $a+b=1/4$, $ab=0$ ($a=0$, $b=1/4$, $c=3/4$).

In this case $\hat{\lambda}+\hat{\mu}+\hat{\nu}=1$ and hypergeometric equation (21), derived from (16), is integrable with these a, b, c.

Case 3 : $c=\frac{3}{4}$, $a+b=\frac{1}{4}$, $ab=-\frac{1}{8}\frac{1}{1+2e}$ $\left(a,\ b=\left(\frac{1}{4}\pm\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}}\right)\frac{1}{2},\ c=\frac{3}{4}\right)$

In this case $\hat{\lambda}=\frac{1}{4}$, $\hat{\mu}=\frac{1}{2}$, $\hat{\nu}=\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}}$. These values are not in the Table. Therefore, if hypergeometric equation (21) is integrable then condition (i) in the [Theorem] must be stisfied. Thus at least one of

$$(a)\ \hat{\lambda}+\hat{\mu}+\hat{\nu}=\frac{3}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}},\quad (b)\ -\hat{\lambda}+\hat{\mu}+\hat{\nu}=\frac{1}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}},$$

$$(c)\ \hat{\lambda}-\hat{\mu}+\hat{\nu}=-\frac{1}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}},\quad (d)\ \hat{\lambda}+\hat{\mu}-\hat{\nu}=\frac{3}{4}+\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}}$$

must be an odd integer.

Case 4: $c=\frac{3}{4},\ a+b=\frac{1}{4},\ ab=-\frac{1}{8}\frac{1}{1-2e}\ \left(a,\ b=\left(\frac{1}{4}\pm\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}}\right)\frac{1}{2},\ c=\frac{3}{4}\right)$

In this case $\hat{\lambda}=\frac{1}{4},\ \hat{\mu}=\frac{1}{2},\ \hat{\nu}=\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}}$. These values are not in the Table. Therefore, if hypergeometric equation (21) is integrable then condition (i) in the [Theorem] must be stisfied. Thus at least one of

$$(a')\ \hat{\lambda}+\hat{\mu}+\hat{\nu}=\frac{3}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}},\quad (b')\ -\hat{\lambda}+\hat{\mu}+\hat{\nu}=\frac{1}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}},$$

$$(c')\ \hat{\lambda}-\hat{\mu}+\hat{\nu}=-\frac{1}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}},\quad (d')\ \hat{\lambda}+\hat{\mu}-\hat{\nu}=\frac{3}{4}+\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}}$$

must be an odd integer.

Case	$\pm\hat{\lambda}$	$\pm\hat{\mu}$	$\pm\hat{\nu}$	Comment
1	$\frac{1}{2}+l$	$\frac{1}{2}+m$	Arbitrary	
2	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	
3	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$even
4	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{4}+n$	
5	$\frac{2}{3}+l$	$\frac{1}{4}+m$	$\frac{1}{4}+n$	$l+m+n=$even
6	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	
7	$\frac{2}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$even
8	$\frac{2}{3}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	$l+m+n=$even
9	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{5}+n$	$l+m+n=$even
10	$\frac{3}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	$l+m+n=$even
11	$\frac{2}{5}+l$	$\frac{2}{5}+m$	$\frac{2}{5}+n$	$l+m+n=$even
12	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$even
13	$\frac{4}{5}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	$l+m+n=$even
14	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	$l+m+n=$even
15	$\frac{3}{5}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	$l+m+n=$even

Table of Schwarz-Hukuhara-Ohasi

Consequently, the following theorem is proved.

[**Theorem 1**] If $\epsilon=3/2$ then hypergeometric equation (21) is integrable independently from e. If $\epsilon=\frac{1}{2}$ then hypergeometric equation (21) is integrable for e which at least one of (a), (b), (c), (d) gives ann odd integer and at least one of (a′), (b′), (c′), (d′) gives an odd integer.

(Proof) This theorem is proved by above discussion. □

By Morales-Luiz, Ramis [4] and Kimura's [3] following theorem is proved:

[**Theorem 2**] If $\epsilon=3/2$ then Hamilton system (1) may be integrable independently from e. If there does not exist e which at least one of (a), (b), (c), (d) gives an odd integer and at least one of (a′), (b′), (c′), (d′) gives an odd integer then Hamilton system (1) must not be integrable, for $\epsilon=1/2$.

(Proof) This theorem is proved by Morales-Luiz-Ramis and [Theorem 1]. □

Discussion

In this paper we considered the integrability of nonlinear dynamical system by use of the Kimura's theorem. We show the detail as following. If we study the integarablity of hypergeometric equation (21), then we must consider the existence of e which (a) and (a′) give an odd integer respectively, for example. Then following equations must hold for some integer x, y:

$$\frac{3}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}}=1+2x, \quad \frac{3}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}}=1+2y.$$

By eliminating e from these equations we have

$$y(1-2x-8x^2)(1+4y)=x(-1+2y+8y^2)(1+4x).$$

This is an indeteminate equation for integers x, y. We do not know the existence of solution (x, y) which x, y are integers at this time. But by computing we found that there dose not exist any solution in $-100\leq x, y\leq 100$.

If there is an e which (a) and (b′) gives odd integer, then one has following equation:

$$\frac{3}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1+2e}}=1+2x, \quad \frac{1}{4}-\sqrt{\frac{1}{16}+\frac{1}{2}\frac{1}{1-2e}}=1+2y.$$

By eliminating e from these equations we have

$$(1+6y+8y^2)(-1+2x+8x^2)+4y(3+4y)(1+4x)=0$$

This is an indeteminate equation for integers x, y. We do not know the existence of solution (x, y) which x, y are integers at this time. But by computing we found that there dose not exist any solution in $-100\leq x, y\leq 100$. There are 16 pairs of $\{((a), (a')), ((a), (b')), \cdots, ((d), (d'))\}$. And for each pair there is an indeterminate equation as above. But we find that there dose not exisit solution (x, y), x and y are integers, in $-100\leq x, y\leq 100$.

These results suport the consequence which we arived at in Ishii [2]. By the [Theorem 2], the Hamiltonian system (1) may be integrable for $e=3/2$. But the expectation of integrability of this Hamiltonian system became smaller, for $e=1/2$.

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