ON THE SPRAY GEOMETRIC OBJECTS IN RIEMANNIAN GEOMETRY

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ABSTRACT

In the last twenty years many geometers, [1, 2, 3, 5], have studied, with remarkable results, the notion of spray in tangent bundles. It is a fundamental one since the geometry of the total space of tangent bundle can be based on the concept of spray.

We introduced the so-called "S-geometric object field", starting from a spray on tangent bundle. It is determined by the system of local coefficients of the considered spray. In the paper [2] we investigate the general theory of Lie derivarive of this object field and some applications.

Here, in the present paper we develop the theory of a S-geometric object field in the Riemannian spaces. In this case appear new properties of its Lie derivative, as for instance : the group of invariance of this geometric object is the extension to tangent bundle of the group of affine motions of the Riemannian space. Also, we study this object with respect to the isometries of the Riemannian space or with respect to its conformal transformations [3].

In the following we use notations and preliminary results from our papers [2, 3].

§ 1. Sprays on *TM*.

Let M be a real n-dimensional smooth manifold. We assume that all geometric object fields considered in the present paper are of the class C^∞.

We denote by (TM, π, M) the tangent bundle of the manifold M and for any point $u \in TM$ we take the canonical coordinates (x^i, y^i), $(i, j, k, \ldots = 1, \ldots, n)$.

Let us consider also the Liouville vector field on TM :

$$\Gamma = y^i \frac{\partial}{\partial y^i} \tag{1.1}$$

and the natural tangent structure $J : \mathfrak{X}(TM) \to \mathfrak{X}(TM)$, [5]:

As we know, a spray on TM is a vector field $S \in \mathfrak{X}(TM)$ with the property

$$JS = \Gamma \tag{1.2}$$

Locally S can be uniquely expressed by the following form :

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i} \tag{1.3}$$

The system of functions $G^i(x, y)$ give us the coefficients of the spray S.

The integral curves of a spray S are given by the system of differential equations :

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0, \tag{1.4}$$

which has a geometrical meaning. Namely, if (1. 4) is given and it is preserved by the transformations of local coordinates on TM, then $G^i(x, y)$ are the coefficients of a spray on TM.

One shows, [5] that : If the base manifold M is paracompact, then on TM there exist the sprays.

The following results were established in my papers [2,3] :

Theorem 1.1. *The system of coefficients* $\{G^i(x, y)\}$ *of a spray S determines a linear nonhomogeneous geometric object field on TM.*

This is reason because it is called a Spray geometric object field or a S-geometric object field.

A local vector field $\xi^i(x)$ on the manifold M defines on TM the following infinitesimal transformation :

$$\begin{aligned} 'x^i &= x^i + \xi^i(x)\, du, \\ 'y^i &= y^i + y^m \partial_m \xi^i(x)\, du, \end{aligned} \tag{1.5}$$

where du is an infinitesimal constant.

The operator of Lie derivation with respect to (1. 5) will be denoted by $\pounds_\xi$

We have [2, 3] :

Theorem 1. 2. *The Lie derivative* $\pounds_\xi G^i$ *of a S-geometric object field* $G^i(x, y)$ *is given by :*

$$\pounds_\xi G^i = \gamma_\xi G^i - G^j \partial_j \xi^i + \frac{1}{2} y^j y^k \partial_j \partial_k \xi^i \tag{1.6}$$

where γ_ξ *is the operator :*

$$\gamma_\xi = \xi^j \partial_j + y^j \partial_j \xi^m \dot{\partial}_m \tag{1.6'}$$

From this follows that :

a. $\pounds_\xi G^i$ is a d-vector field on TM.

b. If $u^i(x, y)$ is a vector field, then $G^i + u^i$ is a S-geometric object field and we have :

$$\pounds_\xi (G^i + u^i) = \pounds_\xi G^i + \pounds_\xi u^i, \tag{1.7}$$

where

$$\pounds_\xi u^i = \gamma_\xi h^i - u^m \partial_m \xi^i \tag{1.8}$$

In general, [6], the Lie derivative $\pounds_\xi$ of a d-tensor field, for instance for $T^i_j(x, y)$, is as follows :

(1.9) $$\pounds_{\xi} T^i_j = \gamma_{\xi} T^i_j - T^m_j \partial_m \xi^i + T^i_m \partial_j \xi^m.$$

We can give :

Definition 1. 1. *We say that* (1. 5) *is an infinitesimal gauge transformation of the S-geometric object field* $G^i(x, y)$ *if it has the property :*

(1. 10) $$\pounds_{\xi} G^i(x, y) = 0.$$

Therefore we can apply the theory of the Lie derivative of geometric objects, defined on *TM*, studied in the papers [2, 3, 6].

§ 2. *S*-geometric objects in the Riemannian geometry.

Let $\mathcal{R}^n = (M^n, g(x))$ be a Riemann space. On *TM* its "energy function" is given by

(2. 1) $$E(x, y) = g_{ij}(x) y^i y^j.$$

The Euler-Lagrange equations of the Lagrangian $E(x, y)$ are as follows :

$$\frac{\partial E}{\partial x^i} - \frac{d}{dt}\frac{\partial E}{\partial y^i} = 0, \quad (y^i = \frac{dx^i}{dt}).$$

They can be set in the equivalent form

(2. 2) $$\frac{d^2 x^i}{dt^2} + \left\{ {i \atop jk} \right\}(x) \frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

Consequently, we have :

Theorem 2. 1. *The set of functions*

(2. 3) $$G^i(x, y) = \frac{1}{2}\left\{ {i \atop jk} \right\}(x) y^j y^k$$

is a S-geometric object field on TM, which depend on the metric $g_{ij}(x)$ *of the space* $\mathcal{R}^n$, only.

Proof. Of course, the system of differential equations (2. 2) has a geometrical meaning. So, comparing (2. 2) with (1. 4) we obtain (2. 3)

By virtue of this theorem we say that $\{G^i(x,y)\}$ from (2. 3) is the canonical *S*-geometric object field of the space $\mathcal{R}^n$.

Remarking that in the case when the parameter t from (2. 2) is the arc-length, then (2. 2) are the equations of geodesics in $\mathcal{R}^n$, it follows :

Proposition 2. 1. *The equations of geodesics in* $\mathcal{R}^n$ *uniquely determines the canonical S-geometric object field of the space* $\mathcal{R}^n$.

Proposition 2. 2. *The vector field* $\xi^i(x)$ *is a Killing vector field of the Riemannian space*

$\mathcal{R}^n$ *if and only if it has the property* :

$$\pounds_\xi E(x, y) = 0.$$

Indeed, from (2. 1) we deduce $\pounds_\xi E(x, y) = (\pounds_\xi g_{ij}(x))y^i y^j$,

So, $\pounds_\xi E(x, y) = 0$ is equivalent to the Killing equations $\pounds_\xi g_{ij}(x) = 0$.

Let " $|$ " be the operator of covariant derivative with respect to the Levi-Civita connection of the space $\mathcal{R}^n$ and $R^i_{jkh}(x)$ its curvature tensor field.

The following result belongs to the author :

Theorem 2. 2. *For a Riemannian space* $\mathcal{R}^n$ *we have* :

1°. *The Lie derivative* $\pounds_\xi$ *of the canonical S-geometric object field* $G^i(x, y)$ *can be expressed in the form* :

$$\pounds_\xi G^i(x, y) = \frac{1}{2} y^j y^k (\xi^i_{|j|k} + \xi^m R^i_{jkm}). \tag{2. 4}$$

2° or in the equivalent form

$$\pounds_\xi G^i(x, y) = \frac{1}{2} (\pounds_\xi \left\{ {i \atop jk} \right\}(x))\, y^j y^k. \tag{2. 5}$$

Therefore we can formulate :

Theorem 2. 3. *If the vector field* ξ^i *is a Killing vector field of the space* $\mathcal{R}^n$, *then then it is an infinitesimal gauge transformation of the canonical S-geometric object field of this space.*

Theorem 2. 4. *The infinitesimal transformation* ξ^i *is a gauge transformation for the canonical S-geometric object field of the space* $\mathcal{R}^n$, *if and only if it is an infinitesimal affine motion of the space* $\mathcal{R}^n$.

Theorem 2. 5. *The group of gauge transformation (if it exists) of the canonical S-geometric object field of the Riemannian space* $\mathcal{R}^n$ *is the prolongation to tangent bundle of the group of affine motions of the space* $\mathcal{R}^n$.

Proof. The second part of Theorem 2 .2 implies that the equations

$$\pounds_\xi G^i(x, y) = 0 \text{ and } \pounds_\xi \left\{ {i \atop jk} \right\}(x) = 0$$

are equivalent. Indeed, from $(\pounds_\xi \left\{ {i \atop jk} \right\}(x))\, y^j y^k = 0$, differentiating with respect to y^i we get the mentioned property. The previous equivalence shows that Theorem holds.

§ 3. Conformal transformations in $\mathcal{R}^n$ and the S-geometric objects.

Let us consider a conformal change of metric in the Riemannian space $\mathcal{R}^n = (M^n, g_{ij}(x))$:

(3. 1) $$g^*_{ij}(x) = e^{2\sigma(x)} g_{ij}(x).$$

As we know, (3. 1) implies the transformation of Christoffel symbols of the metrics g_{ij} and g^*_{ij} as follows :

(3. 2) $$\left\{ {i \atop jk} \right\}^* = \left\{ {i \atop jk} \right\} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - g_{jk} g^{ih} \sigma_h (\sigma_i = \partial_i \sigma).$$

Therefore the canonical S-geometric object field

$$G^{*i}(x, y) = \frac{1}{2} \left\{ {i \atop jk} \right\}^* (x) y^j y^k$$

is related to the canonical S-geometric object field $G^i(x,y)$ from (2.3) by the formula

(3. 3) $$G^{*i}(x, y) = G^i(x, y) + u^i(x, y)$$

where u^i is the following d-vector field :

(3. 4) $$u^i(x, y) = y^i(\sigma_m y^m) - \frac{1}{2} E(x, y) g^{im} \sigma_m .$$

It is convenient to set :

(3. 5) $$\varphi^{jk}(x, y) = y^i y^j - \frac{1}{2} E(x, y) g^{ij}$$

Therefore, $\varphi^{ij}(x, y)$ is a d-tensor field, expressed by means of the energy function $E(x, y)$ and by the metric of the space $\mathcal{R}^n$.

We can write :

(3. 6) $$u^i(x, y) = \varphi^{ij}(x, y) \sigma_j(x).$$

Proposition 3. 1. *The d-tensor field* $\varphi^{ij}(x, y)$ *is non-singular.*

Indeed, the following d-tensor

$$\varphi^*_{ij} = \frac{4}{E^2} \{ y_i y_j - \frac{1}{2} E g_{ij} \}, (y_i = g_{ij}(x) y^j),$$

has the propery $\varphi^*_{ij} \varphi^{jk} = \delta^k_i$

By virtue of the previous results it follows :

Theorem 3. 1. *The canonical S-geometric object field* $G^i(x, y)$ of the *space* $\mathcal{R}^n$ *is invariant with respect to a conformal change* (3. 1) *if and only if* (3. 1) *is a homomothety.*

The d-tensor φ^{ij} has an interesting property.

Proposition 3. 2. *If the infinitesimal transformation is an isometry of the space* $\mathcal{R}^n$, *then the Lie derivative* $\pounds_\xi \varphi^{ij}(x, y)$ *vanishes.*

Proof. Applying the the Proposition 2.2 , it follows that the infinitesimal isometry ξ^i and the fact that for any infinitesimal transformation we have $\pounds_\xi y^i = 0$ imply $\pounds_\xi \varphi^{ij} = 0$.

An infinitesimal transformation $\xi^i(x)$ is called conformal if it has the property

$$\pounds_\xi g_{ij} = 2\sigma g_{ij} \tag{3. 7}$$

So, we have more general result as in the previous Proposition :

Theorem 3. 2. *For any infinitesimal conformal transformation* ξ^i *on TM we have*

$$\pounds_\xi \varphi^{ij}(x, y) = 0. \tag{3. 8}$$

Proof. Because $\pounds_\xi y^i = 0$ and from (3. 7) follows $\pounds_\xi g^{ij} = -2\sigma g^{ij}$, from (3. 5) we get (3. 8) by the straight calculation.

Therefore, by means of (3. 6) we deduce :

Proposition 3. 3. *The Lie derivative of the d-vector field* $u^i(x, y)$ *is given by :*

$$\pounds_\xi u^i = \varphi^{ij} \pounds_\xi \sigma_j \tag{3. 9}$$

Proposition 3. 4. *The following property holds*

$$\pounds_\xi u^i = 0 \longleftrightarrow \pounds_\xi \sigma_j = 0 \tag{3. 10}$$

This property follows from the last three Propositions.

Now, taking into account the previous properties we can prove the following theorems :

Theorem 3. 3. *We have*

$$\pounds_\xi G^{*i}(x, y) = \pounds_\xi G^i(x, y)$$

if and only if $\pounds_\xi \sigma_j = 0$

Theorem 3. 4. *The infinitesimal conformal transformations* ξ^i *which are infinitesimal gauge conformal transformations for the canonical S-geometric object field* $G^i(x, y)$ *of the space* $\mathcal{R}^n$ *have the same property for the S-geometric object field* $G^{*i}(x, y)$ *if and only if we have* $\pounds_\xi \sigma_j = 0$.

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