KOWALEVSKI'S EXPONENTS OF TWO OR THREE DIMENSIONAL PERTURVED YANG-MILLS EQUATIONS

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Abstract

In this paper we consider the perturved 2- or 3-dimensional Yang-Mills equation. These equations are nonlinear and similarity invariant. By use of these property we investigate the Kowalevski's exponent and integrability of perturved Yang-Mills equations.

1. Introduction

In this article, inorder to look for the integrable dynamical system we consider the perturved Yang-Mills equations.

In this article, we call the equation (2.1) perturved 2-dimension Yang-Mills equation. This system is derived from the perturved Hamiltonian

$$H_1(q_1, q_2, p_1, p_2\,;\ e)=\frac{1}{2}(p_1^2+p_2^2)+e(q_1^4+q_2^4)+(q_1^2q_2^2), \tag{1.1}$$

where unperturved Hamiltonian which gives the 2-dimensional Yang-Mills equation is

$$\widetilde{H}(q_1, q_2, p_1, p_2)=\frac{1}{2}(p_1^2+p_2^2)+(q_1^2q_2^2).$$

Our Hamiltonian system derived from (1.1) is similarity invariant system. An integrable similarity invariant system derived from a Hamiltonian of type of (1.2)

$$H_0(q_1, q_2, p_1, p_2)=\frac{1}{2}(p_1^2+p_2^2)+V(q_1, q_2) \tag{1.2}$$

must be algebraically integrable (see [1]). We call that the similarity invariant system is algebraically integrable when it has sufficiently many algebraic integrals. Yoshida [1] has proved that the similarity invariant system is algebraically integrable then it's Kowalevski's exponent (KE) must be rational. In other words, if a similarity invariant system derived from Hamiltonian (1.2) has irrational KE then it is not algebraically integrable and furthermore not integrable.

By calculation we can show that unperturved 2-dimensional and 3-dimensional Yang-Mills equations have nonreal KEs. These KEs shall be shown later. Thus they are not integrable (see [1], [5]).

By use of this result, we consider the integrability of perturbed Yang-Mills equation.

2. Two-dimensional Yang-Mills equation

Consider the perturved system of Yang-Mills equations

$$\begin{cases} \dfrac{dq_1}{dt} = F_1(q_1, q_2, p_1, p_2\,;\ e) = p_1, \\ \dfrac{dq_2}{dt} = F_2(q_1, q_2, p_1, p_2\,;\ e) = p_2, \\ \dfrac{dp_1}{dt} = F_3(q_1, q_2, p_1, p_2\,;\ e) = -2q_1q_2^2 - 4eq_1^3, \\ \dfrac{dp_2}{dt} = F_4(q_1, q_2, p_1, p_2\,;\ e) = -2q_2q_1^2 - 4eq_2^3, \end{cases} \tag{2.1}$$

where e is a parameter. This system is derived from the Hamiltonian (1.1) and it is invariant under the transformation

$$t \to \alpha^{-1}t,\ q_1 \to \alpha^1 q_1,\ q_2 \to \alpha^1 q_2,\ p_1 \to \alpha^2 p_1,\ p_2 \to \alpha^2 p_2. \tag{2.2}$$

Thus we call the system (2.1) similarity invariant under the transformation (2.2). Where $(g_1, g_2, g_3, g_4) = (1, 1, 2, 2)$ is given uniquely by the system of linear equations (2.4) (see [1]) :

$$\sum_{j=1}^{4}\left[x_j \frac{\partial F_i}{\partial x_j}(x_1, x_2, x_3, x_4\,;\ e)\right] = (g_i + 1)F_i(x_1, x_2, x_3, x_4\,;\ e) \quad (i = 1, 2, 3, 4) \tag{2.3}$$

where one put $(q_1, q_2, p_1, p_2) = (x_1, x_2, x_3, x_4)$ in $F_i(q_1, q_2, p_1, p_2\,;\ e)$ of (2.1).

For the similarity invariant system (2.1) there are particular solutions

$$(q_1(t), q_2(t), p_1(t), p_2(t)) = (c_1t^{-1}, c_2t^{-1}, c_3t^{-2}, c_4t^{-2}), \tag{2.4}$$

where c_1, c_2, c_3, c_4 are constants which will be determined later. One can show that $(q_1(t), q_2(t), p_1(t), p_2(t))$ satisfies the invariant system (2.1) (see [1]). The constnts c_1, c_2, c_3, c_4 are determined by system of nonlinear equations

$$-g_ic_i = F_1(c_1, c_2, c_3, c_4\,;\ e) \quad (i = 1, 2, 3, 4). \tag{2.5}$$

The equation (2.5) has 9 solutions.

Inorder to have the KE we take the variational equation around the above particular solution. The variational equation is following

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{4} \frac{\partial F_i}{\partial x_j}(c_1t^{-1}, c_2t^{-1}, c_3t^{-2}, c_4t^{-2}\,;\ e)\xi_j \quad (i = 1, 2, 3, 4). \tag{2.6}$$

One can rewrite above equation in following :

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{4} \frac{\partial F_i}{\partial x_j}(c_1, c_2, c_3, c_4\,;\ e)t^{g_j - g_i - 1}\xi_j \quad (i = 1, 2, 3, 4). \tag{2.6$'$}$$

This system has solution

$$(\xi_1, \xi_2, \xi_3, \xi_4)=(\xi_{1,0}t^{\rho-g_1}, \xi_{2,0}t^{\rho-g_2}, \xi_{3,0}t^{\rho-g_3}, \xi_{4,0}t^{\rho-g_4}), \tag{2.7}$$

where $(\xi_{1,0}, \xi_{2,0}, \xi_{3,0}, \xi_{4,0})^T$ is an eigenvector correspondent to eigenvalue ρ of

$$K=\left(\frac{\partial F_i}{\partial x_j}(c_1, \cdots, c_4\,;\ e)+\delta_{ij}g_i\right) \quad (i, j=1, 2, 3, 4).$$

By direct calculation one has

$$K=\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -12ec_1^2-2c_2^2 & -4c_1c_2 & 2 & 0 \\ -4c_1c_2 & -12ec_2^2-2c_1^2 & 0 & 2 \end{pmatrix}. \tag{2.8}$$

And corresponding to each solution (c_1, c_2, c_3, c_4) there is a set of KE. Thus there are 9 sets and they are classified into 3-classes. There is a class $\{1, 1, 2, 2\}$ for $(c_1, c_2, c_3, c_4)=(0, 0, 0, 0)$ among 3 classes. This trivial case is removed from our consideration. Following classes will be considered :

$$A_1=\left\{-1, 4, \frac{3\sqrt{e}\pm\sqrt{4+e}}{2\sqrt{e}}\right\},$$

$$A_2=\left\{-1, 4, \frac{3+6e\pm\sqrt{-7+36e+100e^2}}{2(1+2e)}\right\}.$$

Theorem 1 *The matrix K has 4 eigen values KE, corresponding to each (c_1, c_2, c_3, c_4). The sets of KE is classified into 2 classes A_1, A_2.*

Proof. Omitted. □

Inorder to look for e which gives rational KE, one must consider the function $f(e)=\sqrt{4+e}/2\sqrt{e}$, $g(e)=\sqrt{-7+36e+100e^2}/2(1+2e)$. One takes $e=1/2$, $1/6$ which gives rational $f(1/2)=g(1/2)=3/2$, $f(1/6)=5/2$, $g(1/6)=1/2$. By the [Theorem 1], for $e=1/2$, $1/6$ all KEs of the system (2.1) are rational. It is necessary to investigate the integrability of (2.1) in these case.

Make a scale transformation in (2.1) $q_1=kQ_1$, $q_2=kQ_2$, $p_1=kP_1$, $p_2=kP_2$ then one has a system

$$\begin{cases} \dfrac{dQ_1}{dt}=P_1, \\ \dfrac{dQ_2}{dt}=P_2, \\ \dfrac{dP_1}{dt}=-2k^2Q_1Q_2^2-4k^2eQ_1^3, \\ \dfrac{dP_2}{dt}=-2k^2Q_2Q_1^2-4k^2eQ_2^3. \end{cases} \tag{2.9}$$

When $e=1/2$, put $k=\sqrt{1/2}$ then (2.9) becomes the Hamiltonian system which derived from

$$H(Q_1, Q_2, P_1, P_2\,;\ 1)=\frac{1}{2}(P_1^2+P_2^2)+\frac{1}{4}(Q_1^4+Q_2^4)+\frac{1}{2}Q_1^2Q_2^2. \tag{2.10}$$

When $e=1/6$, put $k=\sqrt{3/2}$ then (2.9) becomes the Hamiltonian system which derived from

$$H(Q_1, Q_2, P_1, P_2\,;\ 3)=\frac{1}{2}(P_1^2+P_2^2)+\frac{1}{4}(Q_1^4+Q_2^4)+\frac{3}{2}Q_1^2Q_2^2. \tag{2.10$'$}$$

It is known that the Hamiltonian systems derived from the (2.10) and (2.10′) are integrable (See [2]). And furthermore, for only $\epsilon=0, 1, 3$ the Hamiltonian

$$H(Q_1, Q_2, P_1, P_2\,;\ \epsilon)=\frac{1}{2}(P_1^2+P_2^2)+\frac{1}{4}(Q_1^4+Q_2^4)+\frac{\epsilon}{2}Q_1^2Q_2^2. \tag{2.11}$$

gives integrable Hamiltonian systems. Thus, there dose not exist any integrable Hamiltonian system (2.9) derived from (2.11) except $e=1/2$, $1/6$, and one has following theorem.

Theorem 2 *For only $e=1/2$, $1/6$ the perturved 2-dimensional Yang-Mills equation (2.1) are integrable.*

Proof. Corresponding to parameter $\epsilon=1$ there is the first integral $\Phi_1=P_1Q_2-P_2Q_1$ besides $H(Q_1, Q_2, P_1, P_2\,;\ 1)$. There is the first integral $\Phi_2=P_1P_2+Q_1Q_2(Q_1^2+Q_2^2)$ besides $H(Q_1, Q_2, P_1, P_2\,;\ 3)$ corresponding to $\epsilon=3$. By use of these functions the first integral besides the Hamiltonian $H_1(q_1, q_2, p_1, p_2\,;\ e)$ are given as following. For $e=1/2$ there is the first integral $\Psi_1=2(p_1q_2-p_2q_1)$ beside the $H_1(q_1, q_2, p_1, p_2\,;\ 1/2)$. And, for $e=1/6$ there is the first integral $\Psi_2=(2/3)p_1p_2+(4/9)(q_1q_2)(q_1^2+q_2^2)$ besides $H_1(q_1, q_2, p_1, p_2\,;\ 1/6)$. But, it is necessary to show that Ψ_1 is indedendent from $H_1(q_1, q_2, p_1, p_2\,;\ 1/2)$ and Ψ_2 is independent from $H_1(q_1, q_2, p_1, p_2\,;\ 1/6)$. In the first place one notice that $H_1(q_1, q_2, p_1, p_2\,;\ e)$ is homogeneous polynomial of weighted degree 4. The integral Ψ_1 is a homogeneous polynomial of weighted degree 3. Thus Ψ_1 is independent from $H_1(q_1, q_2, p_1, p_2\,;\ 1/2)$. On the other hand, Ψ_2 is a homogeneous polynomial of weighted degree 4. But, one can show that vector $\nabla\Psi_2(0, 0, 1, 0)$ is linearly idependent from $\nabla H_1(0, 0, 1, 0\,;\ 1/6)$. Thus Ψ_2 is the first integral independent from $H_1(q_1, q_2, p_1, p_2\,;\ 1/6)$ (see [1]).

By above discussion for $e=1/2$, $1/6$ the perturved 2-dimensional Yang-Mills equations are integrabel. Theorem was proved. □

3. Three-dimensional Yang-Mills equation

The unperturved 3-dimensional Yang-Mills equation is derived from the following Hamiltonian

$$\widetilde{H}(q_1, q_2, q_3, p_1, p_2, p_3)=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+V(q_1, q_2, q_3),$$
$$V(q_1, q_2, q_3)=q_1^2q_2^2+q_2^2q_3^2+q_3^2q_1^2.$$

Consider the perturved Hamiltonian

$$
\begin{aligned}
&H_2(q_1, q_2, q_3, p_1, p_2, p_3;\ e)=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+V(q_1, q_2, q_3),\\
&V(q_1, q_2, q_3;\ e)=e(q_1^4+q_2^4+q_3^4)+(q_1^2q_2^2+q_2^2q_3^2+q_3^2q_1^2),
\end{aligned}
\tag{3.1}
$$

where e is a parameter. The perturved 3-dimensional Yang-Mills equation which derived from (3.1) is as following

$$
\begin{cases}
\dfrac{dq_1}{dt}=F_1(q_1, q_2, q_3, p_1, p_2, p_3;\ e)=p_1,\\
\dfrac{dq_2}{dt}=F_2(q_1, q_2, q_3, p_1, p_2, p_3;\ e)=p_2,\\
\dfrac{dq_3}{dt}=F_3(q_1, q_2, q_3, p_1, p_2, p_3;\ e)=p_3,\\
\dfrac{dp_1}{dt}=F_4(q_1, q_2, q_3, p_1, p_2, p_3;\ e)=-2q_1(q_2^2+q_3^2)-4eq_1^3,\\
\dfrac{dp_2}{dt}=F_5(q_1, q_2, q_3, p_1, p_2, p_3;\ e)=-2q_2(q_1^2+q_3^2)-4eq_2^3,\\
\dfrac{dp_3}{dt}=F_6(q_1, q_2, q_3, p_1, p_2, p_3;\ e)=-2q_3(q_1^2+q_2^2)-4eq_3^3.
\end{cases}
\tag{3.2}
$$

This system is similarity invariant under the transformation

$$
\begin{aligned}
&t\to \alpha^{-1}t,\ q_1\to \alpha^1 q_1,\ q_2\to \alpha^1 q_2,\ q_3\to \alpha^1 q_3,\\
&p_1\to \alpha^2 p_1,\ p_2\to \alpha^2 p_2,\ p_3\to \alpha^2 p_3.
\end{aligned}
\tag{3.3}
$$

The set of constants $(g_1, g_2, g_3, g_4, g_5, g_6)=(1, 1, 1, 2, 2, 2)$ is given by

$$
\sum_{j=1}^{6}\left[x_j\frac{\partial F_i}{\partial x_j}(x_1, \cdots, x_6;\ e)\right]=(g_i+1)F_i(x_1, \cdots, x_6;\ e)\quad (i=1, \cdots, 6).
\tag{3.4}
$$

And the similarity invariant system (3.3) has particular solutins

$$
(q_1, q_2, q_3, p_1, p_2, p_3)=(c_1t^{-1}, c_2t^{-1}, c_3t^{-1}, c_4t^{-2}, c_5t^{-2}, c_6t^{-2}).
\tag{3.5}
$$

The constnts $c_1, c_2, c_3, c_4, c_5, c_6$ are determined, as solutions, by the system of nonlinear equations

$$
-g_ic_i=F_i(c_1, c_2, c_3, c_4, c_5, c_6;\ e)\quad (i=1, 2, \cdots, 6),
\tag{3.6}
$$

where $(g_1, g_2, g_3, g_4, g_5, g_6)=(1, 1, 1, 2, 2, 2)$. The equation (3.6) has 27 solutions. Each solution is a set of 6 components of complex number. Thus the system (3.2) has 27 particular solutions. Around each particular solution, one can construct the variational equation

$$
\frac{d\xi_i}{dt}=\sum_{j=1}^{6}\frac{\partial F_i}{\partial x_j}(c_1t^{-1}, c_2t^{-1}, \cdots, c_6t^{-2};\ e)\xi_j\quad (i=1, \cdots, 6).
\tag{3.7}
$$

This variational equation has a solution

$$
(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)=(\xi_{1,0}t^{\rho-g_1}, \xi_{2,0}t^{\rho-g_2}, \xi_{3,0}t^{\rho-g_3}, \xi_{4,0}t^{\rho-g_4}, \xi_{5,0}t^{\rho-g_5}, \xi_{6,0}t^{\rho-g_6}),
\tag{3.8}
$$

where $(\xi_{1,0}, \xi_{2,0}, \xi_{3,0}, \xi_{4,0}, \xi_{5,0}, \xi_{6,0})^T$ is an eigen vector corresponding to eigenvalue ρ of K:

$$K=\left(\frac{\partial F_i}{\partial x_j}(c_1, \cdots, c_6;\ e)+\delta_{ij}g_i\right) \quad (i, j=1, 2, 3, 4, 5, 6). \tag{3.9}$$

By direct calculation one has K as following:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -12ec_1^2-2(c_2^2+c_3^2) & -4c_1c_2 & -4c_1c_3 & 2 & 0 & 0 \\ -4c_1c_2 & -12ec_2^2-2(c_1^2+c_3^2) & -4c_2c_3 & 0 & 2 & 0 \\ -4c_1c_3 & -4c_2c_3 & -12ec_3^2-2(c_1^2+c_2^2) & 0 & 0 & 2 \end{bmatrix}.$$

Corresponding to each set $(c_1, c_2, c_3, c_4, c_5, c_6)$ there is a set of eigen values $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6)$. Thus there are 27-sets of eigen values. Each eigen values is the KE of (3.2).

27-sets of KE are classified into 4-classes. There is a class $\{1, 1, 1, 2, 2, 2\}$ for $(c_1, c_2, c_3, c_4, c_5, c_6)=(0, 0, 0, 0, 0, 0)$ among 4 classes. This trivial case is removed from our consideration. Following classes will be considered:

$$B_1=\left\{-1, 4, \frac{3\sqrt{e}\pm\sqrt{4+e}}{2\sqrt{e}}, \frac{3\sqrt{e}\pm\sqrt{4+e}}{2\sqrt{e}}\right\},$$

$$B_2=\left\{-1, 4, \frac{3+6e\pm\sqrt{17+36e+4e^2}}{2(1+2e)}, \frac{3+6e\pm\sqrt{-7+36e+100e^2}}{2(1+2e)}\right\}$$

$$B_3=\left\{-1, 4, \frac{3+3e\pm\sqrt{1+26e+25e^2}}{2(1+e)}, \frac{3+3e\pm\sqrt{1+26e+25e^2}}{2(1+e)}\right\}.$$

Inorder to look for e which gives rational KE one solves the equation

$$\frac{\sqrt{17+36e+4e^2}}{1+2e}=x, \quad \frac{\sqrt{-7+36e+100e^2}}{1+2e}=y,$$

where, it is enough to consider in the domain $x>0$, $y>0$. Then one has the solutions $e=8/(x^2-1)-1/2$, $e=16/(25-y^2)-1/2$. From this relation one has the indeterminate equation $2x^2+y^2=27$. One can take rational solutions $(x_0, y_0)=(3, 3)$, $(x_1, y_1)=(1, 5)$, $(x_2, y_2)=(11/3, 1/3)$. For each solution one has a parameter $e_0=1/2$, $e_1=\infty$, $e_2=1/7$. In our investigation parameter e should not be infinity, we exclude this case of $e_1=\infty$. We take $e_0=1/2$. In this case we have $B_i=\{-1, 4, 3, 0, 3, 0\}$ $(i=1, 2, 3)$. In this case all KEs of the perturved 3-dimensional Yang-Mills equation are integers. For $e_2=1/7$ there is an irrational KE $(3+\sqrt{29})/2$ in B_1. In this case the perturved 3-dimensional Yang-Mills equation has irrational KEs.

By above investigation one has following theorem.

Theorem 3 *The perturved 3-dimensional Yang-Mills equation (3.1) has KEs of integers for $e=1/2$. For $e=1/7$ Yang-Mills equation (3.1) has irrational KEs and furthermore it must not be integrable in this case.*

Proof. Omitted. □

Discussion In this article we considered the existence of rational KE and integrability of perturved Yang-Mills equations which are similarity invariant.

The first result is that 2-dimensional perturved Yang-Mills equations are integrable for $e=1/2, 1/6$. And it is interesting that the 2-dimensional unperturved Yang-Mills equation must not be integrable while some 2-dimensional perturved Yang-Mills equations are integrable.

The nonintegrability of 2-dimensional unperturved Yang-Mills equation is proved by the fact that it has nonreal KEs $C=\{-1, 4, (3\pm\sqrt{-7})/2\}$.

The second result is as following. Every classes of KEs of the 3-dimensional perturved Yang-Mills equation have only elements of integers for $e=1/2$, and they have elements of irrational numbers for $e=1/7$. And it may be integrable in the first case, and it must not be integrable in the second case (see [1], [2], [4]). But the above indeteminate equation is not solved completely. The author has obtained 30 solutions of that indeterminate equation. Among these solutions, only $(x_0, y_0)=(3, 3)$ gives classes of KEs with only rational nunmbers (integers).

The unperturved 3-dimensioal Yang-Mills equation has nontribial 20-classes of KEs. And thay are classified into 2 clases

$$D_1=\{-1, 4, (3\pm i\sqrt{7})/2, (3\pm\sqrt{17})/2\}, \quad D_2=\{-1, 4, 1, 1, 2, 2\}.$$

Thus unperturved 3-dimensional Yang-Mills equation is not integrable (see [3], [5]).

It is interesting that the perturved Yang-Mills equation (3.1) has KEs of only integers for parameter $e=1/2$ while 3-dimensional unperturved Yang-Mills equation must has irrational KEs. And it should be remarked that (3.1) may be integrable for $e=1/2$.

At the end, the author notice that we used Mathematica to solve simultaneous algebraic equations of heigher degree and the eigen equation of large Kowalevski's matrix.

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