Integrability of Hamilton System with Three Dimensional Perturved Yang-Mills Potential and Gauss Hypergeometric Equation

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Abstract

Abstract In Ishii [1, 2] we considered Hamilton systems with perturved Yang-Mills potential. This potential contains a parameter e. We showed that, for some parametrs, there were sets of integral Kowalevskii's exponents (see [8]). But we could not show the existence of enough many integrals for integrability of this Hamiltonian even if $e=2$. Therefore we must chracterize the parameter e more definitely.

M. Lakshmanan and R. Sahadevan [5] had considered some Hamilton system and they had given two aditional integrals besides the Hamiltonian. We showed the uniqueness of the parameter e ($|e|<\infty$) which gave the aditional two integrals besides the Hamiltonian introduced by M. Lakshmanan and R. Sahadevan. In this paper we apply the Theorem by T. Kimura as we did in [3] to characterize e.

1. Introduction

We follow Yoshida [9] and we call

$$H(q_1,q_2,q_3,p_1,p_2,p_3)=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+V_0(q_1,q_2,q_3), \tag{1}$$

$$V_0(q_1,q_2,q_3)=(q_1^2q_2^2+q_2^2q_3^2+q_3^2q_1^2) \tag{2}$$

the Hamiltonian with the Yang-Mills potential $V_0(q_1,q_2,q_3)$. And we call

$$H(q_1,q_2,q_3,p_1,p_2,p_3)=T(p_1,p_2,p_3)+V(q_1,q_2,q_3), \tag{3}$$

$$T(p_1,p_2,p_3)=\frac{1}{2}(p_1^2+p_2^2+p_3^2), \tag{4}$$

$$V(q_1,q_2,q_3)=\frac{e}{4}(q_1^4+q_2^4+q_3^4)+(q_1^2q_2^2+q_2^2q_3^2+q_3^2q_1^2) \tag{5}$$

the Hamiltonian with the perturved Yang-Mills potential $V(q_1,q_2,q_3)$, where e is a parameter.

It is known that the Hamilton system (1) is not integrable, by computing the Kowalevskii's exponents of (1), because (1) has irrational Kowalevskii's exponents.

On the other hand, Hamilton system (3) has Kowalevskii's exponents with only integers for only $e=1/2, \infty$ (see Ishii[1]). This is a necessary condition for the integrability of (3). It is interesting to chracterize the parameter e in detail from another point of view.

In the first place we consider the integrability of (3) when $e \neq \infty$. After these con sideration, we study the integrability of (3) under the condition $e=\infty$.

2. Straight-line solutions and variational equations around them

We write $(q_1,q_2,q_3)=\boldsymbol{q}$, $(p_1,p_2,p_3)=\boldsymbol{p}$ and we consider canonical equations:

$$\frac{d}{dt}\boldsymbol{q}=\mathbf{grad}\ \boldsymbol{T}(\boldsymbol{p}),\ \frac{d}{dt}\boldsymbol{p}=-\mathbf{grad}\ \boldsymbol{V}(\boldsymbol{q}). \tag{6}$$

The straight-line solution is given by

$$\boldsymbol{q}=\boldsymbol{c}Q(t),\ \boldsymbol{p}=\boldsymbol{c}P(t),\ \boldsymbol{c}=(c_1,c_2,c_3), \tag{7}$$

where $\boldsymbol{c}$ is determined by the algebraic equation

$$\boldsymbol{c}=\mathbf{grad}\ \boldsymbol{T}(\boldsymbol{c})=\mathbf{grad}\ \boldsymbol{V}(\boldsymbol{c}). \tag{8}$$

and $Q(t)$, $P(t)$ will be given later as a solution of the simultaneuos diffierential equation

$$\frac{d}{dt}Q=P,\ \frac{d}{dt}P=-Q^3. \tag{9}$$

By solving the equation (8), involving a trivial solution $(0,0,0)=\boldsymbol{c}_0$ one has 27 solutions $\boldsymbol{c}_j\,(j=0, 1, 2, \cdots, 26)$. Thus by use of these solutions of (8) one has straight-line solutions except trivial case. In this discussion we consider nontrivial cases only.

At each straight-line solution one has a variational equation (VE) :

$$\frac{d}{dt}\boldsymbol{\xi}=D^2\boldsymbol{T}(\boldsymbol{c}_j)\,\boldsymbol{\eta},\ \frac{d}{dt}\boldsymbol{\eta}=-Q(t)^2D^2\boldsymbol{V}(\boldsymbol{c}_j)\,\boldsymbol{\xi}\quad(j=1, 2, \cdots, 26), \tag{10}$$

where $D^2\boldsymbol{T}(\boldsymbol{c}_j)$, $D^2\boldsymbol{V}(\boldsymbol{c}_j)$ are Hessian at $\boldsymbol{p}=\boldsymbol{c}_j$, $\boldsymbol{q}=\boldsymbol{c}_j$, and $\boldsymbol{\xi}=\delta\boldsymbol{q}$, $\boldsymbol{\eta}=\delta\boldsymbol{p}$ respectively.

By computing, concretely, for $\boldsymbol{c}_j(\boldsymbol{j}=1, 2, \cdots, 6)$ the eigenvalues of $D^2\boldsymbol{V}(\boldsymbol{c}_j)$ are $\lambda_{1,1}=2/e$, $\lambda_{1,2}=2/e$, $\lambda_{1,3}=3$, for $\boldsymbol{c}_j(j=7, 8, \cdots, 18)$ the eigenvalues of $D^2\boldsymbol{V}(\boldsymbol{c}_j)$ are $\lambda_{2,1}=4/(e+2)$, $\lambda_{2,2}=(-2+3e)/(e+2)$, $\lambda_{2,3}=3$, and for $\boldsymbol{c}_j(j=19, 20, \cdots, 26)$ the eigenvalues of $D^2\boldsymbol{V}(\boldsymbol{c}_j)$ are $\lambda_{3,1}=3e/(e+4)$, $\lambda_{3,2}=3\boldsymbol{e}/(\boldsymbol{e}+4)$, $\lambda_{3,3}=3$. Notice that by use of homogeneity of $V(q_1,q_2,q_3)$, $\lambda_{i,i}=3\,(i=1, 2, 3)$.

Notice the symmetry of $D^2\boldsymbol{V}(\boldsymbol{c}_j)$ then one has following VEs by change of variable:

$$(VE_1)\quad \boldsymbol{\xi}''=-\boldsymbol{Q}(t)^2 Diag(\lambda_{1,1},\lambda_{1,2},\lambda_{1,3})\,\boldsymbol{\xi}\quad(\boldsymbol{c}_j,\ j=1, 2, \cdots, 6),$$
$$(VE_2)\quad \boldsymbol{\xi}''=-\boldsymbol{Q}(t)^2 Diag(\lambda_{2,1},\lambda_{2,2},\lambda_{2,3})\,\boldsymbol{\xi}\quad(\boldsymbol{c}_j,\ j=7, 8, \cdots, 18),$$
$$(VE_3)\quad \boldsymbol{\xi}''=-\boldsymbol{Q}(t)^2 Diag(\lambda_{3,1},\lambda_{3,2},\lambda_{3,3})\,\boldsymbol{\xi}\quad(\boldsymbol{c}_j,\ j=19, \cdots, 26).$$

3. Solvability of Gauss Hypergeometric equation

$P=P(t)$, $Q=Q(t)$, the solution of differential equation (9), is given as following.

The equation (9) is derived by one-degree-of-freedom Hamiltonian

$$h=P^2+\frac{1}{4}Q^4. \tag{11}$$

If we set the value $h=\frac{1}{4}$, we find that $Q=Q(t)$ is given by the inverse function of

$$t-t_0=\int\left[\frac{(1-Q^4)}{2}\right]^{-1/2}dQ. \tag{12}$$

By use of $Q=Q(t)$ we have $P=P(t)$ and equation (9) is solved completely.

Now we explain the normal and tangential variational equation by use of (VE_1).

$$\xi_3''=-Q(t)^2\lambda_{1,3}\xi_3. \tag{13}$$

is called the tangential variational equation. On the other hand the equation

$$\boldsymbol{\xi}''=-Q(t)^2 Diag(\lambda_{1,1},\lambda_{1,2})\boldsymbol{\xi} \tag{14}$$

is called the normal variational equation (NVE) for $\boldsymbol{\xi}=(\xi_1,\xi_2)$.

Consequently, the NVEs are following :

$$\begin{aligned}
&(NVE_1) \qquad \boldsymbol{\xi}''=-Q(t)^2 Diag(\lambda_{1,1},\lambda_{1,2})\boldsymbol{\xi},\\
&(NVE_2) \qquad \boldsymbol{\xi}''=-Q(t)^2 Diag(\lambda_{2,1},\lambda_{2,2})\boldsymbol{\xi},\\
&(NVE_3) \qquad \boldsymbol{\xi}''=-Q(t)^2 Diag(\lambda_{3,1},\lambda_{3,2})\boldsymbol{\xi}.
\end{aligned}$$

One has the following Gauss hypergeometric equations from $NVEs$ by transformation of independent variable $z=Q(t)^4$.

$$(ANVE_1) \quad \begin{cases} z(1-z)\dfrac{d^2}{dz^2}\xi_1+\left(\dfrac{3}{4}-\dfrac{5}{4}z\right)\dfrac{d}{dz}\xi_1+\dfrac{\lambda_{1,1}}{8}\xi_1=0, & (ANVE_{1,1})\\ z(1-z)\dfrac{d^2}{dz^2}\xi_2+\left(\dfrac{3}{4}-\dfrac{5}{4}z\right)\dfrac{d}{dz}\xi_2+\dfrac{\lambda_{1,2}}{8}\xi_2=0. & (ANVE_{1,2}) \end{cases}$$

$$(ANVE_2) \quad \begin{cases} z(1-z)\dfrac{d^2}{dz^2}\xi_1+\left(\dfrac{3}{4}-\dfrac{5}{4}z\right)\dfrac{d}{dz}\xi_1+\dfrac{\lambda_{2,1}}{8}\xi_1=0, & (ANVE_{2,1})\\ z(1-z)\dfrac{d^2}{dz^2}\xi_2+\left(\dfrac{3}{4}-\dfrac{5}{4}z\right)\dfrac{d}{dz}\xi_2+\dfrac{\lambda_{2,2}}{8}\xi_2=0. & (ANVE_{2,2}) \end{cases}$$

$$(ANVE_3) \quad \begin{cases} z(1-z)\dfrac{d^2}{dz^2}\xi_1+\left(\dfrac{3}{4}-\dfrac{5}{4}z\right)\dfrac{d}{dz}\xi_1+\dfrac{\lambda_{3,1}}{8}\xi_1=0, & (ANVE_{3,1})\\ z(1-z)\dfrac{d^2}{dz^2}\xi_2+\left(\dfrac{3}{4}-\dfrac{5}{4}z\right)\dfrac{d}{dz}\xi_2+\dfrac{\lambda_{3,2}}{8}\xi_2=0. & (ANVE_{3,2}) \end{cases}$$

$ANVE_1$, $ANVE_2$, $ANVE_3$ are systems of independent Gauss hypergeometric equations and they are the algebraic normal variational equations. Then each $ANVE_j$ is given by

$$ANVE_j=ANVE_{j,1} \oplus ANVE_{j,2} (j=1,\,2,\,3).$$

These are direct sums. It is known that the $ANVE_j$ is solvable, if and only if, each $ANVE_{j,i}(i=1,\,2)$ is solvable (see [7]). First of all we consider the coefficients of above Gauss hypergeometric equations as following.

Generally, Gauss hypergeometric equation is given by

$$z(1-z)\frac{d^2}{dz^2}\xi+\{c-(a+b+1)z\}\frac{d}{dz}\xi-ab\xi=0. \tag{15}$$

In our case a, b, c are given as following (see Yoshida [10]) :

$$a+b=\frac{1}{2}-\frac{1}{4},\ ab=-\frac{\lambda_{i,j}}{8},\ c=1-\frac{1}{4}\quad (i=1,2,3\,;\ j=1,2). \tag{16}$$

Constants a, b, c are determined as bellow :

$$a=\frac{1}{8}\left\{1-\sqrt{\frac{e+16}{e}}\right\},\ b=\frac{1}{8}\left\{1+\sqrt{\frac{e+16}{e}}\right\},\ c=\frac{3}{4}\left(\lambda_{1,1},\lambda_{1,2}=\frac{2}{e}\right), \tag{17}$$

$$a=\frac{1}{8}\left\{1-\sqrt{\frac{e+34}{e+2}}\right\},\ b=\frac{1}{8}\left\{1+\sqrt{\frac{e+34}{e+2}}\right\},\ c=\frac{3}{4}\left(\lambda_{2,1}=\frac{4}{e+2}\right), \tag{18}$$

$$a=\frac{1}{8}\left\{1-\sqrt{\frac{25e-14}{e+2}}\right\},\ b=\frac{1}{8}\left\{1+\sqrt{\frac{25e-14}{e+2}}\right\},\ c=\frac{3}{4}\left(\lambda_{2,2}=\frac{-2+3e}{e+2}\right), \tag{19}$$

$$a=\frac{1}{8}\left\{1-\sqrt{\frac{25e+4}{e+4}}\right\},\ b=\frac{1}{8}\left\{1+\sqrt{\frac{25e+4}{e+4}}\right\},\ c=\frac{3}{4}\left(\lambda_{3,1},\lambda_{3,2}=\frac{3e}{e+4}\right). \tag{20}$$

Then, we take up the [Theorem (T. Kimura)] to study the solvability of $(ANV\ E_1)$, $(ANVE_2)$ and $(ANVE_3)$ (see [4]).

K is the field of the set of all rational functions, the field which is obtained from K by the adjunction of the solutions of the linear ordinary differential equation considered, is called the Picard-Vessiot extension of K. The field L, which is obtained from K by a number of steps each of which is either a finite algebraic extension or the adjunction of an indefinite integral or the adjunction of an exponential of an indefinite integral, is called the generarized Liouville extension.

The Riemann's P function of hypergeometric Equation (15) is written in the following form

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix}. \tag{21}$$

Let the difference of exponents $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$ at $z=0,1,\infty$ then $\hat{\lambda}=1-c$, $\hat{\mu}=c-a-b$, $\hat{\nu}=b-a$.

Kimura's theorem is stated as following (see H. Yoshida [11]).

[**Theorem (T. Kimura)**]([**4**], [**10**]) Let L be the Picard-Vessiot extension of K for Gauss hypergeometric equation (15). Inorder to be a generalized Liouville extension of K, it is necessary and sufficient that either

(i) at least one of $\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $-\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $\hat{\lambda}-\hat{\mu}+\hat{\nu}$, $\hat{\lambda}+\hat{\mu}-\hat{\nu}$ is an odd integer, or

(ii) $\pm\hat{\lambda}$, $\pm\hat{\mu}$, $\pm\hat{\nu}$ take values in Table, called the Table of Schwarz-Hukuhara-Ohasi, in an arbitrary order, with integer l, m, n.

Case	$\pm\hat{\lambda}$	$\pm\hat{\mu}$	$\pm\hat{\nu}$	Comment
1	$\frac{1}{2}+l$	$\frac{1}{2}+m$	Arbitrary	
2	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	
3	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$even
4	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{4}+n$	
5	$\frac{2}{3}+l$	$\frac{1}{4}+m$	$\frac{1}{4}+n$	$l+m+n=$even
6	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	
7	$\frac{2}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$even
8	$\frac{2}{3}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	$l+m+n=$even
9	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{5}+n$	$l+m+n=$even
10	$\frac{3}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	$l+m+n=$even
11	$\frac{2}{5}+l$	$\frac{2}{5}+m$	$\frac{2}{5}+n$	$l+m+n=$even
12	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$even
13	$\frac{4}{5}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	$l+m+n=$even
14	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	$l+m+n=$even
15	$\frac{3}{5}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	$l+m+n=$even

(Table of Schwarz-Hukuhara-Ohasi)

4. Integrability of Hamilton system with Yang-Mills potential

By T. Kimura's theorem we characterize the parameter e which gives integrability of Hamilton system (3). By computation one has following :

[**Theorem 1**] Every Gauss hypergeometric equations ($ANVE_{i,j}(i=1, 2, 3\,;\ j=1, 2)$) are solvable if and only if $e=2$.

(Proof) For every finite e, $\pm\hat{\lambda}$, $\pm\hat{\mu}$, $\pm\hat{\nu}$ do not take values in Table of Shwarz-Hukuhara-Ohasi. To prove this theorem, consider $a+b=1/4$, $ab=\lambda$ then, we have solutions $a=(1-\sqrt{1+8\lambda})/8$, $b=(1+\sqrt{1+8\lambda})/8$.

Consider the equations :

$$\hat{\lambda}+\hat{\mu}+\hat{\nu}=\frac{3+\sqrt{1+8\lambda}}{4}=2k+1, \tag{E1}$$

$$-\hat{\lambda}+\hat{\mu}+\hat{\nu}=\frac{1+\sqrt{1+8\lambda}}{4}=2k+1, \tag{E2}$$

$$\hat{\lambda}-\hat{\mu}+\hat{\nu}=\frac{-1+\sqrt{1+8\lambda}}{4}=2k+1, \tag{E3}$$

$$\hat{\lambda}+\hat{\mu}-\hat{\nu}=\frac{3-\sqrt{1+8\lambda}}{4}=-2k+1, \tag{E4}$$

where k is nonnegative integer.

Put $\lambda=2/e$ and solve the equations (E_i) of e then one has $e=F_{i,1}(k)$ such as $\lim_{k\to\infty} F_{i,1}(k)=0$. Furtheremore $F_{1,1}(0)=\infty$, $0.2=F_{1,1}(1)>F_{1,1}(k)>0\,(k\geq 1)$ and $F_{4,1}(0)=-\infty$, $1/3=F_{4,1}(1)>F_{4,1}(k)>0\,(k\geq 1)$ and $2/3=F_{3,1}(0)>F_{3,1}(k)>0\,(k\geq 1)$. Notice that $2=F_{2,1}(0)>F_{2,1}(k)>0\,(k\geq 1)$ especially.

Put $\lambda=4/(e+2)$ and solve the equations (E_i) of e then one has $e=F_{i,2}(k)$ such as $\lim_{k\to\infty} F_{i,2}(k)=-2\,(i=1,2,3,4)$. Furthermore $F_{1,2}(0)=\infty$, $-1.6\geq F_{1,2}(k)>-2\,(k\geq 1)$ and $-2/3=F_{3,2}(0)>F_{3,2}(k)>-2\,(k\geq 1)$ and $F_{4,2}(0)=-\infty$, $-3/4=F_{4,2}(1)>F_{4,2}(k)>-2\,(k\geq 1)$. Notice that $F_{2,2}(0)=2$ and $-26/15=F_{2,2}(1)>F_{2,2}(k)>-2\,(k\geq 1)$ especially.

Put $\lambda=(3e-2)/(e+2)$ and solve the equations (E_i) of e then one has $e=F_{i,3}(k)$ such as $\lim_{k\to\infty} F_{i,3}(k)=-2\,(i=1,2,3,4)$. Furthermore $F_{1,3}(0)=2/3$, $-22/7=F_{1,3}(1)\leq F_{1,3}(k)<-2\,(k\geq 1)$ and $F_{3,3}(0)=\infty$, $-22/9=F_{3,3}(1)\leq F_{3,3}(k)<-2\,(k\geq 1)$ and $F_{4,3}(0)=2/3$, $-14/3=F_{4,3}(1)\leq F_{4,3}(k)<-2\,(k\geq 1)$. Notice that $F_{2,3}(0)=2$ and $-8/3=F_{2,3}(1)<F_{2,3}(k)<-2\,(k\geq 1)$ especially.

Put $\lambda=3e/(e+4)$ and solve the equations (E_i) of e then one has $e=F_{i,4}(k)$ such as $\lim_{k\to\infty} F_{i,4}(k)=-4\,(i=1,2,3,4)$. Furthermore $F_{i,4}(0)=0$, $-40/7=F_{1,4}(1)\leq F_{1,4}(k)<-4\,(k\geq 1)$ and $F_{3,4}(0)=\infty$, $-14/3=F_{3,4}(1)\leq F_{3,4}(k)<-4\,(k\geq 1)$ and $F_{4,4}(0)=0$, $-8=F_{4,4}(1)\leq F_{4,4}(k)<-4\,(k\geq 1)$. Notice that $F_{2,4}(0)=2$, $-5=F_{2,4}(1)\leq F_{2,4}(k)<-4\,(k\geq 1)$ especially.

To determine the parameter e which at least one of $\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $-\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $\hat{\lambda}-\hat{\mu}+\hat{\nu}$, $\hat{\lambda}+\hat{\mu}-\hat{\nu}$ gives an odd integer at each eigenvalue $\lambda_{1,1}=\lambda_{1,2}$, $\lambda_{2,1}$, $\lambda_{2,2}$, $\lambda_{3,1}=\lambda_{3,2}$, one must determine nonnegative integers k_1,k_2,k_3,k_4 such as $F_{i(1),1}(k_1)=F_{i(2),2}(k_2)=F_{i(3),3}(k_3)=F_{i(4),4}(k_4)$.

Conversely, if one has integers k_1,k_2,k_3,k_4 such as $F_{i(1),1}(k_1)=F_{i(2),2}(k_2)=F_{i(3),3}(k_3)=F_{i(4),4}(k_4)=e$, then $\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $-\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $\hat{\lambda}-\hat{\mu}+\hat{\nu}$, $\hat{\lambda}+\hat{\mu}-\hat{\nu}$ gives an odd integer at each above eigenvalue for this e.

By above discussion one can determine the parameter $e=2$ with above property uniquely. Indeed for $e=2$ we have

$$-\hat{\lambda}+\hat{\mu}+\hat{\nu}=-\frac{1}{4}+\frac{1}{2}+\frac{3}{4}=1 \quad \left(\lambda_{1,1},\ \lambda_{1,2}=\frac{2}{e}\right), \tag{22}$$

$$-\hat{\lambda}+\hat{\mu}+\hat{\nu}=-\frac{1}{4}+\frac{1}{2}+\frac{3}{4}=1 \quad \left(\lambda_{2,1}=\frac{4}{e+2}\right), \tag{23}$$

$$-\hat{\lambda}+\hat{\mu}+\hat{\nu}=-\frac{1}{4}+\frac{1}{2}+\frac{3}{4}=1 \quad \left(\lambda_{2,2}=\frac{-2+3e}{e+2}\right), \tag{24}$$

$$-\hat{\lambda}+\hat{\mu}+\hat{\nu}=-\frac{1}{4}+\frac{1}{2}+\frac{3}{4}=1 \quad \left(\lambda_{3,1},\ \lambda_{3,2}=\frac{3e}{e+4}\right), \tag{25}$$

and for another e, there exists an eigenvalue that none of $\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $-\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $\hat{\lambda}-\hat{\mu}+\hat{\nu}$, $\hat{\lambda}+\hat{\mu}-\hat{\nu}$ gives an odd integer at the eigenvalue.

Thus this theorem is proved and $ANVE_{i,j}\,(i=1,2,3\,;\ j=1,2)$ are solvable if and only if $e=2$. Consequently $ANE_i(i=1, 2, 3)$ are all solvable if and only if $e=2$. □

Morales-Luiz and Ramis [6] showed following (see [6, 7, 11]).

[**Theorem (Morales-Luiz and Ramis)**] If a Hamilton system is integrable then the normal variational equations around the each straight-line solutions must be solvable. And, the normal variational equation is solvable, if and only if Gauss Hypergeometric equations corresponding to normal variational equations around the straight-line solutions are solvable.

(Proof) Omitted (See [6, 7, 11]). □

By use of above [Theorem (Morales-Luiz and Ramis)] we have following theorem.

[**Theorem 2**] If the Hamilton system (3) is integrable for e, then e must be 2.

(Proof) If (3) is integrable then all normal variational equations $NVE_i(i=1, 2, 3)$ are solvable, by [Theorem (Morales-Luiz and Ramis)]. Therefore $ANVE_i(i=1, 2, 3)$ are solvable and $ANVE_{i,j}(i=1, 2, 3; j=1, 2)$ are solvable. Thus one has $e=2$ by [Theorem 1]. Thus theorem is proved. □

On the other hand, Lakshimanan and Shadevan considered the Hamilton system (see [5]).

$$H_3=\frac{1}{2}\{p_1^2+p_2^2+p_3^2\}+Aq_1^2+Bq_2^2+Cq_3^2+\alpha q_1^4+\beta q_2^4+\gamma q_3^4+\delta q_1^2q_2^2+\epsilon q_2^2q_3^2+\omega q_3^2q_1^2 \qquad (26)$$

They gave the following two integrals I_1, I_2 besides H_3 for $A=B=C$, $\alpha=\beta=\gamma$, $2\alpha=\delta=\epsilon=\omega$:

$$I_1= q_1p_2-q_2p_1,$$
$$I_2=(q_1p_2-q_2p_1)^2+(q_2p_3-q_3p_2)^2+(q_3p_1-q_1p_3)^2.$$

Thus the Hamilton system H_3 is integrable in this case. For $e=2$, Hamilton system (3) satisfys these conditions, by Lakshimanan and Sahadevan, and it is integrable. Thus we have following theorem.

[**Theorem 3**] Hamilton system (3) with three dimensional perturved Yang-Mills potential is integrable if and only if $e=2$.

(Proof) By above discussion the proof is stated. Therefore the proof is omitted. □

5. A perturvation when $e=\infty$

We considered the integrability of the perturved Hamilton system (3) for $|e|<\infty$ in the previous section. In the discussion of [1], we knew that $e=\infty$ gave an integral Kowalevskii's exponent. This parameter $e=\infty$ may give the integrability of the Hamilton system (3). But one dose not have the definition about the perturvation for $e=\infty$. Therefore we define the perturvation when $e=\infty$ in the first place.

By a transformation $p_i=\tau P_i, q_i=\tau Q_i (i=1, 2, 3)$ we have a system of equations

$$Q_i' = P_i \ (i=1,\ 2,\ 3)$$
$$P_1' = -e\tau^2 Q_1^3 - 2\tau^2\{Q_1 Q_2^2 + Q_2^2 Q_1\}$$
$$P_2' = -e\tau^2 Q_2^3 - 2\tau^2\{Q_1^2 Q_2 + Q_2 Q_3^2\}$$
$$P_3' = -e\tau^2 Q_3^3 - 2\tau^2\{Q_2^2 Q_3 + Q_3 Q_1^2\}.$$

Put $\tau = 1/\sqrt{e}$ and $e \to \infty$, then we have a system of equations

$$Q_i' = P_i, P_i' = -Q_i^3 \ (i=1,\ 2,\ 3).$$

This is a system of equations derived from

$$H^*(Q_1,Q_2,Q_3,P_1,P_2,P_3) = \frac{1}{2}(P_1^2+P_2^2+P_3^2) + V^*(Q_1,Q_2,Q_3), \tag{27}$$

$$V^*(Q_1,Q_2,Q_3) = \frac{1}{4}(Q_1^4+Q_2^4+Q_3^4). \tag{28}$$

There are two integrals $I_i^* = \frac{P_i^2}{2} + \frac{Q_i^4}{4} \ (i=1,\ 2)$ besides $H^*(Q_1,Q_2,Q_3,P_1,P_2,P_3)$.

Therefore the Hamilton system $H^*(Q_1,Q_2,Q_3,P_1,P_2,P_3)$ is integrable. Let us define the integrability of perturved $H(q_1,q_2,q_3,p_1,p_2,p_3)$ at $e=\infty$ by the integrability of $H^*(Q_1,Q_2,Q_3,P_1,P_2,P_3)$. In this sence we have following theorem.

[**Theorem 4**] The Hamilton system (3)

$$H(q_1,q_2,q_3,p_1,p_2,p_3) = T(p_1,p_2,p_3) + V(q_1,q_2,q_3),$$
$$T(p_1,p_2,p_3) = \frac{1}{2}(p_1^2+p_2^2+p_3^2),$$
$$V(q_1,q_2,q_3) = \frac{e}{4}(q_1^4+q_2^4+q_3^4) + (q_1^2q_2^2+q_2^2q_3^2+q_3^2q_1^2)$$

is integrable when $e=\infty$.

(Proof) This theorem is proved by above discussion. □

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